RESOLUTION OF SINGULARITIES FOR A CLASS OF HILBERT MODULES

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Abstract. A short proof of the "Rigidity theorem" using the sheaf theoretic model for Hilbert modules over polynomial rings is given. The joint kernel for a large class of submodules is described. The completion $[\mathcal{I}]$ of a homogeneous (polynomial) ideal $\mathcal{I}$ in a Hilbert module is a submodule for which the joint kernel is shown to be of the form

$$\{p_i(\tfrac{\partial}{\partial \bar{w}_1}, \ldots, \tfrac{\partial}{\partial \bar{w}_m})K_{[\mathcal{I}]}(\cdot, w)|_{w=0}, 1 \leq i \leq n\},$$

where $K_{[\mathcal{I}]}$ is the reproducing kernel for the submodule $[\mathcal{I}]$ and $p_1, \ldots, p_n$ is some minimal "canonical set of generators" for the ideal $\mathcal{I}$. The proof includes an algorithm for constructing this canonical set of generators, which is determined uniquely modulo linear relations, for homogeneous ideals. A set of easily computable invariants for these submodules, using the monoidal transformation, are provided. Several examples are given to illustrate the explicit computation of these invariants.

1. Preliminaries

Beurling's theorem describing the invariant subspaces of the multiplication (by the coordinate function) operator on the Hardy space of the unit disc is essential to the Sz.-Nagy – Foias model theory and several other developments in modern operator theory. In the language of Hilbert modules, Beurling's theorem says that all submodules of the Hardy module of the unit disc are equivalent. This observation, due to Cowen and Douglas [6], is peculiar to the case of one-variable operator theory. The submodule of functions vanishing at the origin of the Hardy module $H_0^2(\mathbb{D}^2)$ of the bi-disc is not equivalent to the Hardy module $H^2(\mathbb{D}^2)$. To see this, it is enough to note that the joint kernel of the adjoint of the multiplication by the two co-ordinate functions on the Hardy module of the bi-disc is 1 - dimensional (it is spanned by the constant function 1) while the joint kernel of these operators restricted to the submodule is 2 - dimensional (it is spanned by the two functions z_1 and z_2).

There has been a systematic study of this phenomenon in the recent past [1, 10] resulting in a number of "Rigidity theorems" for submodules of a Hilbert module $\mathcal{M}$ over the polynomial ring $\mathbb{C}[\underline{z}] := \mathbb{C}[z_1, \ldots, z_m]$ of the form $[\mathcal{I}]$ obtained by taking the norm closure of a polynomial ideal $\mathcal{I}$ in the Hilbert module. For a large class of polynomial ideals, these theorems often take the form: two submodules $[\mathcal{I}]$ and $[\mathcal{J}]$ in some Hilbert module $\mathcal{M}$ are equivalent if and only if the two ideals $\mathcal{I}$ and $\mathcal{J}$ are equal. We give a short proof of this theorem using the sheaf theoretic model developed earlier in [2] and construct tractable invariants for Hilbert modules over $\mathbb{C}[\underline{z}]$.

Let $\mathcal{M}$ be a Hilbert module of holomorphic functions on a bounded open connected subset Ω of $\mathbb{C}^m$ possessing a reproducing kernel K. Assume that $\mathcal{I} \subseteq \mathbb{C}[\underline{z}]$ is the singly generated ideal $\langle p \rangle$. Then the reproducing kernel $K_{[\mathcal{I}]}$ of $[\mathcal{I}]$ vanishes on the zero set $V(\mathcal{I})$ and the map $w \mapsto K_{[\mathcal{I}]}(\cdot, w)$ defines a holomorphic Hermitian line bundle on the open set $\Omega^*_{\mathcal{I}} = \{w \in \mathbb{C}^m : \bar{w} \in \Omega \setminus V(\mathcal{I})\}$ which naturally extends to all of Ω^*. As is well known, the curvature of this line bundle completely determines the equivalence class of the Hilbert module $[\mathcal{I}]$ (cf. [4, 5]). However, if $\mathcal{I} \subseteq \mathbb{C}[\underline{z}]$ is not a principal ideal, then the corresponding line bundle defined on $\Omega^*_{\mathcal{I}}$ no longer extends to all of Ω^*. Indeed, it was conjectured

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in [8] that the dimension of the joint kernel of the Hilbert module $[\mathcal{I}]$ at w is 1 for points w not in $V(\mathcal{I})$, otherwise it is the codimension of $V(\mathcal{I})$. Assuming that

(a) $\mathcal{I}$ is a principal ideal or
(b) w is a smooth point of $V(\mathcal{I})$.

Duan and Guo verify the validity of this conjecture in [11]. Furthermore if $m = 2$ and $\mathcal{I}$ is prime then the conjecture is valid.

Thus for any submodule $[\mathcal{I}]$ in a Hilbert module $\mathcal{M}$, assuming that $\mathcal{M}$ is in the Cowen-Douglas class $\mathrm{B}_1(\Omega^*)$ and the co-dimension of $V(\mathcal{I})$ is greater than 1, it follows that $[\mathcal{I}]$ is in $\mathrm{B}_1(\Omega^*_{\mathcal{I}})$ but it doesn't belong to $\mathrm{B}_1(\Omega^*)$. For example, $H^2_0(\mathbb{D}^2)$ is in the Cowen-Douglas class $\mathrm{B}_1(\mathbb{D}^2 \setminus \{(0,0)\})$ but it does not belong to $\mathrm{B}_1(\mathbb{D}^2)$. To systematically study examples of submodules like $H^2_0(\mathbb{D}^2)$, the following definition from [2] will be useful.

Definition. A Hilbert module $\mathcal{M}$ over the polynomial ring in $\mathbb{C}[\underline{z}]$ is said to be in the class $\mathfrak{B}_1(\Omega^*)$ if

(rk) possess a reproducing kernel K (we don't rule out the possibility: $K(w,w) = 0$ for w in some closed subset X of Ω) and
(fin) The dimension of $\mathcal{M}/\mathfrak{m}_w\mathcal{M}$ is finite for all $w \in \Omega$.

For Hilbert modules in $\mathfrak{B}_1(\Omega)$, from [2], we have:

Lemma. Suppose $\mathcal{M} \in \mathfrak{B}_1(\Omega^*)$ is the closure of a polynomial ideal $\mathcal{I}$. Then $\mathcal{M}$ is in $\mathrm{B}_1(\Omega^*)$ if the ideal $\mathcal{I}$ is singly generated while if it is minimally generated by more than one polynomial, then $\mathcal{M}$ is in $\mathrm{B}_1(\Omega^*_{\mathcal{I}})$.

This Lemma ensures that to a Hilbert module in $\mathfrak{B}_1(\Omega^*)$, there corresponds a holomorphic Hermitian line bundle defined by the joint kernel for points in $\Omega^*_{\mathcal{I}}$. We will show that it extends to a holomorphic Hermitian line bundle on the "blow-up" space $\hat{\Omega}^*$ via the monoidal transform under mild hypothesis on the zero set $V(\mathcal{I})$. We also show that this line bundle determines the equivalence class of the module $[\mathcal{I}]$ and therefore its curvature is a complete invariant. However, computing it explicitly on all of $\hat{\Omega}^*$ is difficult. In this paper we find invariants, not necessarily complete, which are easy to compute. One of these invariants is nothing but the curvature of the restriction of the line bundle on $\hat{\Omega}^*$ to the exceptional subset of $\hat{\Omega}^*$.

A line bundle is completely determined by its sections on open subsets. To write down the sections, we use the decomposition theorem for the reproducing kernel [2, Theorem 1.5]. The actual computation of the curvature invariant require the explicit calculation of norm of these sections. Thus it is essential to obtain explicit description of the eigenvectors $K^{(i)}$, $1 \leq i \leq d$, in terms of the reproducing kernel. We give two examples which, we hope, will motivate the results that follow. Let $H^2(\mathbb{D}^2)$ be the Hardy module over the bi-disc algebra. The reproducing kernel for $H^2(\mathbb{D}^2)$ is the Sźego kernel $\mathbb{S}(z,w) = \frac{1}{1-z_1\bar{w}_2}\frac{1}{1-z_2\bar{w}_2}$. Let $\mathcal{I}_0$ be the polynomial ideal $\langle z_1, z_2\rangle$ and let $[\mathcal{I}_0]$ denote the minimal closed submodule of the Hardy module $H^2(\mathbb{D}^2)$ containing $\mathcal{I}_0$. Then the joint kernel of the adjoint of the multiplication operators M_1 and M_2 is spanned by the two linearly independent vectors: $z_1 = p_1(\bar{\partial}_1, \bar{\partial}_2)\mathbb{S}(z,w)_{|w_1=0=w_2}$ and $z_2 = p_2(\bar{\partial}_1, \bar{\partial}_2)\mathbb{S}(z,w)_{|w_1=0=w_2}$, where p_1, p_2 are the generators of the ideal $\mathcal{I}_0$. For a second example, take the ideal $\mathcal{I}_1 = \langle z_1 - z_2, z_2^2\rangle$ and let $[\mathcal{I}_1]$ be the minimal closed submodule of the Hardy module $H^2_0(\mathbb{D}^2)$ containing $\mathcal{I}_1$. The joint kernel is not hard to compute. A set of two linearly independent vectors which span it are $p_1(\bar{\partial}_1, \bar{\partial}_2)\mathbb{S}(z,w)_{|w_1=0=w_2}$ and $p_2(\bar{\partial}_1, \bar{\partial}_2)\mathbb{S}(z,w)_{|w_1=0=w_2}$, where $p_1 = z_1 - z_2$ and $p_2 = (z_1 + z_2)^2$. Unlike the first example, the two polynomials p_1, p_2 are not the generators for the ideal $\mathcal{I}_1$ that were given at the start, never the less, they are easily seen to be a set of generators for the ideal $\mathcal{I}_1$ as well. This prompts the question:

Question: Let $\mathcal{M} \in \mathfrak{B}_1(\Omega^*)$ be a Hilbert module and $\mathcal{I} \subseteq \mathcal{M}$ be a polynomial ideal. Assume without loss of generality that $0 \in V(\mathcal{I})$. We ask

(1) if there exists a set of polynomials $p_1, \ldots, p_n$ such that

$$p_i(\tfrac{\partial}{\partial\bar{w}_1}, \ldots, \tfrac{\partial}{\partial\bar{w}_m})K_{[\mathcal{I}]}(z,w)_{|z=0=w},\ i = 1, \ldots, n,$$

spans the joint kernel of $[\mathcal{I}]$;

(2) what conditions, if any, will ensure that the polynomials $p_1, \ldots, p_n$, as above, is a generating set for $\mathcal{I}$?

We show that the answer to the Question (1) is affirmative, that is, there is a natural basis for the joint eigenspace of the Hilbert module $[\mathcal{I}]$, which is obtained by applying a differential operator to the reproducing kernel $K_{[\mathcal{I}]}$ of the Hilbert module $[\mathcal{I}]$. Often, these differential operators encode an algorithm for producing a set of generators for the ideal $\mathcal{I}$ with additional properties. It is shown that there is an affirmative answer to the Question (2) as well, if the ideal is assumed to be homogeneous. It then follows that, if there were two sets of generators which serve to describe the joint kernel, as above, then these generators must be linear combinations of each other, that is, the sets of generators are determined modulo a linear transformation. We will call them *canonical set of generators*. The canonical generators provide an effective tool to determine if two ideal are equal. A number of examples illustrating this phenomenon is given.

In the following section, we describe the joint kernel. In section 3, we construct the holomorphic Hermitian line bundle on the "blow - up " space. In the last section, we provide an explicit calculation.

1.1. Index of notations:

$\mathbb{C}[\underline{z}]$	the polynomial ring $\mathbb{C}[z_1, \ldots, z_m]$ of m- complex variables
$\mathfrak{m}_w$	maximal ideal of $\mathbb{C}[\underline{z}]$ at the point $w \in \mathbb{C}^m$
Ω	a bounded domain in $\mathbb{C}^m$
Ω^*	$\{\bar{z} : z \in \Omega\}$
$\mathbb{D}$	the open unit disc in $\mathbb{C}$
$\mathbb{D}^m$	the poly-disc $\{z \in \mathbb{C}^m : \lvert z_i\rvert < 1,\ 1 \leq i \leq t\}, m \geq 1$
$[\mathcal{I}]$	the completion of a polynomial ideal $\mathcal{I}$ in some Hilbert module
M_i	module multiplication by the co-ordinate function z_i on $[\mathcal{I}]$, $1 \leq i \leq m$
M_i^*	adjoint of the multiplication operator M_i on $[\mathcal{I}]$, $1 \leq i \leq m$
$K_{[\mathcal{I}]}$	the reproducing kernel of $[\mathcal{I}]$
$\alpha, \lvert\alpha\rvert, \alpha!$	the multi index $(\alpha_1, \ldots, \alpha_m)$, $\lvert\alpha\rvert = \sum_{i=1}^m \alpha_i$ and $\alpha! = \alpha_1! \ldots \alpha_m!$
$\binom{\alpha}{k}$	$= \prod_{i=1}^m \binom{\alpha_i}{k_i}$ for $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $k = (k_1, \ldots, k_m)$
$k \leq \alpha$	if $k_i \leq \alpha_i$, $1 \leq i \leq m$.
z^α	$z_1^{\alpha_1} \ldots z_m^{\alpha_m}$
$\partial^\alpha, \bar{\partial}^\alpha$	$\partial^\alpha = \frac{\partial^{\lvert\alpha\rvert}}{\partial z_1^{\alpha_1} \cdots z_m^{\alpha_m}}, \bar{\partial}^\alpha = \frac{\partial^{\lvert\alpha\rvert}}{\partial \bar{z}_1^{\alpha_1} \cdots \bar{z}_m^{\alpha_m}}$ for $\alpha \in \mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+$
$q(D)$	the differential operator $q(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m})$ ($= \sum_\alpha a_\alpha \partial^\alpha$, where $q = \sum_\alpha a_\alpha z^\alpha$)
$B_n(\Omega)$	Cowen-Douglas class of operators of rank n, $n \geq 1$
q^*	$q^*(z) = \overline{q(\bar{z})}(= \sum_\alpha \bar{a}_\alpha z^\alpha$ for q of the form $\sum_\alpha a_\alpha z^\alpha)$
$\langle\ ,\ \rangle_{w_0}$	the Fock inner product at w_0, defined by $\langle p, q\rangle_{w_0} := q^*(D)p\rvert_{w_0} = (q^*(D)p)(w_0)$
$\mathcal{S}^{\mathcal{M}}$	the analytic subsheaf of $\mathcal{O}_\Omega$, corresponding to the Hilbert module $\mathcal{M} \in \mathfrak{B}_1(\Omega^*)$
$\mathbb{V}_w(\mathcal{F})$	the characteristic space at w, which is $\{q \in \mathbb{C}[\underline{z}] : q(D)f\big\rvert_w = 0$ for all $f \in \mathcal{F}\}$ for some set $\mathcal{F}$ of holomorphic functions

2. Calculation of basis vectors for the joint kernel

The Fock inner product of a pair of polynomials p and q is defined by the rule:

$$\langle p, q\rangle_0 = q^*(\tfrac{\partial}{\partial z_1}, \ldots, \tfrac{\partial}{\partial z_m})\, p\rvert_0,\ q^*(z) = \overline{q(\bar{z})}.$$

The map $\langle\ ,\ \rangle_0 : \mathbb{C}[\underline{z}] \times \mathbb{C}[\underline{z}] \longrightarrow \mathbb{C}$ is linear in first variable and conjugate linear in the second and for $p = \sum_\alpha a_\alpha z^\alpha,\ q = \sum_\alpha b_\alpha z^\alpha$ in $\mathbb{C}[\underline{z}]$, we have

$$\langle p, q\rangle_0 = \sum_\alpha \alpha! a_\alpha \bar{b}_\alpha$$

since $z^\alpha(D)z^\beta|_{z=0} = \alpha!$ if $\alpha = \beta$ and 0 otherwise. Also, $\langle p,p\rangle_0 = \sum_\alpha \alpha!|a_\alpha|^2 \geq 0$ and equals 0 only when $a_\alpha = 0$ for all α. The completion of the polynomial ring with this inner product is the well known Fock space $L^2_a(\mathbb{C}^m, d\mu)$, that is, the space of all μ-square integrable entire functions on $\mathbb{C}^m$, where

$$d\mu(z) = \pi^{-m}e^{-|z|^2}d\nu(z)$$

is the Gaussian measure on $\mathbb{C}^m$ ($d\nu$ is the usual Lebesgue measure).

The characteristic space (cf. [3, page 11]) of an ideal $\mathcal{I}$ in $\mathbb{C}[\underline{z}]$ at the point w is the vector space

$$\mathbb{V}_w(\mathcal{I}) := \{q \in \mathbb{C}[\underline{z}] : q(D)p|_w = 0,\ p \in \mathcal{I}\} \quad = \quad \{q \in \mathbb{C}[\underline{z}] : \langle p, q^*\rangle_w = 0,\ p \in \mathcal{I}\}.$$

The envelope of the ideal $\mathcal{I}$ at the point w is defined to be the ideal

$$\begin{aligned} \mathcal{I}^e_w &:= \{p \in \mathbb{C}[\underline{z}] : q(D)p|_w = 0,\ q \in \mathbb{V}_w(\mathcal{I})\} \\ &= \{p \in \mathbb{C}[\underline{z}] : \langle p, q^*\rangle_w = 0,\ q \in \mathbb{V}_w(\mathcal{I})\}. \end{aligned}$$

It is known [3, Theorem 2.1.1, page 13] that $\mathcal{I} = \cap_{w\in V(\mathcal{I})}\mathcal{I}^e_w$. The proof makes essential use of the well known Krull's intersection theorem. In particular, if $V(\mathcal{I}) = \{w\}$, then $\mathcal{I}^e_w = \mathcal{I}$. It is easy to verify this special case using the Fock inner product. We provide the details below after setting $w = 0$, without loss of generality.

Let $\mathfrak{m}_0$ be the maximal ideal in $\mathbb{C}[\underline{z}]$ at 0. By Hilbert's Nullstellensatz, there exists a positive integer N such that $\mathfrak{m}_0^N \subseteq \mathcal{I}$. We identify $\mathbb{C}[\underline{z}]/\mathfrak{m}_0^N$ with $\mathrm{span}_{\mathbb{C}}\{z^\alpha : |\alpha| < N\}$ which is the same as $(\mathfrak{m}_0^N)^\perp$ in the Fock inner product. Let $\mathcal{I}_N$ be the vector space $\mathcal{I} \cap \mathrm{span}_{\mathbb{C}}\{z^\alpha : |\alpha| < N\}$. Clearly $\mathcal{I}$ is the vector space (orthogonal) direct sum $\mathcal{I}_N \oplus \mathfrak{m}_0^N$. Let

$$\tilde{V} = \{q \in \mathbb{C}[\underline{z}] : \deg q < N \text{ and } \langle p, q\rangle_0 = 0,\ p \in \mathcal{I}_N\} = \left(\mathfrak{m}_0^N\right)^\perp \ominus \mathcal{I}_N.$$

Evidently, $\mathbb{V}_0(\mathcal{I}) = \tilde{V}^*$, where $\tilde{V}^* = \{q \in V : q^* \in \tilde{V}\}$. It is therefore clear that the definition of $\tilde{V}$ is independent of N, that is, if $\mathfrak{m}^{N_1} \subset \mathcal{I}$ for some N_1, then $(\mathfrak{m}_0^{N_1})^\perp \ominus \mathcal{I}_{N_1} = (\mathfrak{m}_0^N)^\perp \ominus \mathcal{I}_N$. Thus

$$\begin{aligned} \mathcal{I}^e_0 &= \{p \in \mathbb{C}[\underline{z}] : \deg p < N \text{ and } \langle p, q^*\rangle_0 = 0,\ q \in \mathbb{V}_0(\mathcal{I})\} \oplus \mathfrak{m}_0^N \\ &= \left((\mathfrak{m}_0^N)^\perp \ominus \tilde{V}\right) \oplus \mathfrak{m}_0^N \\ &= \mathcal{I}_N \oplus \mathfrak{m}_0^N \end{aligned}$$

showing that $\mathcal{I}^e_0 = \mathcal{I}$.

Let $\mathcal{M}$ be a submodule of an analytic Hilbert module $\mathcal{H}$ on Ω such that $\mathcal{M} = [\mathcal{I}]$, closure of the ideal $\mathcal{I}$ in $\mathcal{H}$. It is known that $\mathbb{V}_0(\mathcal{I}) = \mathbb{V}_0(\mathcal{M})$ (cf. [2, 10]). Since

$$\mathcal{M} \subseteq \mathcal{M}^e_0 := \{f \in \mathcal{H} : q(D)f|_0 = 0 \text{ for all } q \in \mathbb{V}_0(\mathcal{M})\},$$

it follows that

$$\begin{aligned} \dim \mathcal{H}/\mathcal{M}^e_0 \leq \dim \mathcal{H}/\mathcal{M} = \dim \mathbb{C}[\underline{z}]/\mathcal{I} &\leq \dim \mathbb{C}[\underline{z}]/\mathfrak{m}_0^N \\ &\leq \sum_{k=0}^{N-1} \binom{k+m-1}{m-1} < +\infty. \end{aligned}$$

Therefore, from [10], we have $\mathcal{M}^e_0 \cap \mathbb{C}[\underline{z}] = \mathcal{I}^e_0$ and $\mathcal{M} \cap \mathbb{C}[\underline{z}] = \mathcal{I}$, and hence

$$\mathcal{M}^e_0 = [\mathcal{I}^e_0] = [\mathcal{I}] = \mathcal{M}. \tag{2.1}$$

Assumption: Let $\mathcal{I} \subseteq \mathbb{C}[\underline{z}]$ be an ideal. We assume that the module $\mathcal{M}$ in $\mathfrak{B}_1(\Omega)$ is the completion of $\mathcal{I}$ with respect to some inner product. For notational convenience, in the following discussion, we let K be the reproducing kernel of $\mathcal{M} = [\mathcal{I}]$, instead of $K_{[\mathcal{I}]}$.

To describe the joint kernel $\cap_{j=1}^m \ker(M_j - w_j)^*$ using the characteristic space $\mathbb{V}_w(\mathcal{I})$, it will be useful to define the auxialliary space

$$\tilde{\mathbb{V}}_w(\mathcal{I}) \ = \ \{q \in \mathbb{C}[\underline{z}] : \frac{\partial q}{\partial z_i} \in \mathbb{V}_w(\mathcal{I}),\ 1 \leq i \leq m\}.$$

From [2, Lemma 3.4], it follows that $V(\mathfrak{m}_w\mathcal{I})\setminus V(\mathcal{I})=\{w\}$ and $\mathbb{V}_w(\mathfrak{m}_w\mathcal{I})=\tilde{\mathbb{V}}_w(\mathcal{I})$. Therefore,

$$\begin{aligned}\dim\cap_{j=1}^m\ker(M_j-w_j)^* &= \dim\mathcal{M}/\mathfrak{m}_w\mathcal{M} \;=\; \dim\mathcal{I}/\mathfrak{m}_w\mathcal{I}\\ &= \sum_{\lambda\in V(\mathfrak{m}_w\mathcal{I})\setminus V(\mathcal{I})}\dim\mathbb{V}_\lambda(\mathfrak{m}_w\mathcal{I})/\mathbb{V}_\lambda(\mathcal{I})\\ &= \dim\tilde{\mathbb{V}}_w(\mathcal{I})/\mathbb{V}_w(\mathcal{I}).\end{aligned}\tag{2.2}$$

For the second and the third equalities, see [3, Theorem 2.2.5 and 2.1.7]. Since $\tilde{\mathbb{V}}_w(\mathcal{I})$ is a subspace of the inner product space $\mathbb{C}[\underline{z}]$, we will often identify the quotient space $\tilde{\mathbb{V}}_w(\mathcal{I})/\mathbb{V}_w(\mathcal{I})$ with the subspace of $\tilde{\mathbb{V}}_w(\mathcal{I})$ which is the orthogonal complement of $\mathbb{V}_w(\mathcal{I})$ in $\tilde{\mathbb{V}}_w(\mathcal{I})$. Equation (2.2) motivates following lemma describing the basis of the joint kernel of the adjoint of the multiplication operator at a point in Ω. This answers the question (1) of the introduction.

Lemma 2.1. *Fix $w_0\in\Omega$ and polynomials $q_1,\ldots,q_t$. Let $\mathcal{I}$ be a polynomial ideal and K be the reproducing kernel corresponding the Hilbert module $[\mathcal{I}]$, which is assumed to be in $\mathfrak{B}_1(\Omega^*)$. Then the vectors*

$$q_1^*(\bar{D})K(\cdot,w)|_{w=w_0},\ldots,q_t^*(\bar{D})K(\cdot,w)|_{w=w_0}$$

form a basis of the joint kernel at w_0 of the adjoint of the multiplication operator if and only if the classes $[q_1],\ldots,[q_t]$ form a basis of $\tilde{\mathbb{V}}_{w_0}(\mathcal{I})/\mathbb{V}_{w_0}(\mathcal{I})$.

Proof. Without loss of generality we assume $0\in\Omega$ and $w_0=0$.

Claim 1: For any $q\in\mathbb{C}[\underline{z}]$, the vector $q^*(\bar{D})K(\cdot,w)|_{w=0}\neq 0$ if and only if $q\notin\mathbb{V}_0(\mathcal{I})$.

Using the reproducing property $f(w)=\langle f,K(\cdot,w)\rangle$ of the kernel K, it is easy to see (cf. [7]) that

$$\partial^\alpha f(w)=\langle f,\bar{\partial}^\alpha K(\cdot,w)\rangle,\ \text{for } \alpha\in\mathbb{Z}_m^+,\ w\in\Omega,\ f\in\mathcal{M}.$$

and thus

$$\begin{aligned}\partial^\alpha f(w)|_{w=0} &= \langle f,\bar{\partial}^\alpha K(\cdot,w)\rangle|_{w=0} \;=\; \langle f,\bar{\partial}^\alpha\{\sum_\beta\frac{\partial^\beta K(z,0)}{\beta!}\bar{w}^\beta\}\rangle|_{w=0}\\ &= \langle f,\{\sum_{\beta\geq\alpha}\frac{\partial^\beta K(z,0)\alpha!}{\beta!}\bar{w}^{\beta-\alpha}\}\rangle|_{w=0} \;=\; \{\sum_{\beta\geq\alpha}\langle f,\frac{\partial^\beta K(z,0)\alpha!}{\beta!}\rangle\bar{w}^{\beta-\alpha}\}|_{w=0}\\ &= \langle f,\bar{\partial}^\alpha K(\cdot,w)|_{w=0}\rangle.\end{aligned}$$

So for $f\in\mathcal{M}$ and a polynomial $q=\sum a_\alpha z^\alpha$, we have

$$\begin{aligned}\langle f,q^*(\bar{D})K(\cdot,w)|_{w=0}\rangle &= \langle q,\sum_\alpha\bar{a}_\alpha\bar{\partial}^\alpha K(\cdot,w)\rangle|_{w=0}=\sum_\alpha a_\alpha\langle f,\bar{\partial}^\alpha K(\cdot,w)\rangle|_{w=0}\\ &= \{\sum_\alpha a_\alpha\partial^\alpha\langle f,K(\cdot,w)\rangle\}|_{w=0}=q(D)f|_{w=0}.\end{aligned}\tag{2.3}$$

This proves the claim.

Claim 2: For any $q\in\mathbb{C}[\underline{z}]$, the vector $q^*(\bar{D})K(\cdot,w)|_{w=0}\in\cap_{j=1}^m\ker M_j^*$ if and only if $q\in\tilde{\mathbb{V}}_0(\mathcal{I})$.

For any $f\in\mathcal{M}$, we have

$$\begin{aligned}\langle f,M_j^*q^*(\bar{D})K(\cdot,w)|_{w=0}\rangle &= \langle M_jf,q^*(\bar{D})K(\cdot,w)|_{w=0}\rangle=q(D)(z_jf)|_{w=0}\\ &= \{z_jq(D)f+\frac{\partial q}{\partial z_j}(D)f\}|_{w=0}=\frac{\partial q}{\partial z_j}(D)f|_{w=0}\end{aligned}$$

verifying the claim.

As a consequence of claims 1 and 2, we see that $q^*(\bar{D})K(\cdot,w)|_{w=0}$ is a non-zero vector in the joint kernel if and only if the class $[q]$ in $\tilde{\mathbb{V}}_0(\mathcal{I})/\mathbb{V}_0(\mathcal{I})$ is non-zero.

Pick polynomials $q_1,\ldots,q_t$. From the equation (2.2) and claim 2, it is enough to show that $q_1^*(\bar D)K(\cdot,w)|_{w=0},\ldots,q_t^*(\bar D)K(\cdot,w)|_{w=0}$ are linearly independent if and only if $[q_1],\ldots,[q_t]$ are linearly independent in $\tilde{\mathbb{V}}_0(\mathcal{I})/\mathbb{V}_0(\mathcal{I})$. But from claim 1 and equation (2.3), it follows that

$$\sum_{i=1}^t \bar\alpha_i q_i^*(\bar D)K(\cdot,w)|_{w=0}=0 \text{ if and only if } \sum_{i=1}^t \alpha_i[q_i]=0 \text{ in } \tilde{\mathbb{V}}_0(\mathcal{I})/\mathbb{V}_0(\mathcal{I})$$

for scalars $\alpha_i\in\mathbb{C}$, $1\le i\le t$. This completes the proof. □

Remark 2.2. The 'if' part of the theorem can also be obtained from the decomposition theorem [2, Theorem 1.5]. For module $\mathcal{M}$ in the class $\mathfrak{B}_1(\Omega^*)$, let $\mathcal{S}^{\mathcal{M}}$ be the subsheaf of the sheaf of holomorphic functions $\mathcal{O}_\Omega$ whose stalk $\mathcal{S}^{\mathcal{M}}_w$ at $w\in\Omega$ is

$$\big\{(f_1)_w\mathcal{O}_w+\cdots+(f_n)_w\mathcal{O}_w : f_1,\ldots,f_n\in\mathcal{M}\big\},$$

and the characteristic space at $w\in\Omega$ is the vector space

$$\mathbb{V}_w(\mathcal{S}^{\mathcal{M}}_w) \;=\; \{q\in\mathbb{C}[\underline{z}] : q(D)f\big|_w=0,\ f_w\in\mathcal{S}^{\mathcal{M}}_w\}.$$

Since

$$\dim\mathcal{S}^{\mathcal{M}}_0/\mathfrak{m}_0\mathcal{S}^{\mathcal{M}}_0=\dim\cap_{j=1}^m\ker M_j^*=\dim\tilde{\mathbb{V}}_0(\mathcal{I})/\mathbb{V}_0(\mathcal{I})=t,$$

there exists a minimal set of generators $g_1,\cdots,g_t$ of $\mathcal{S}^{\mathcal{M}}_0$ and a $r>0$ such that

$$K(\cdot,w)=\sum_{i=1}^t\overline{g_j(w)}K^{(j)}(\cdot,w) \text{ for all } w\in\Delta(0;r)$$

for some choice of anti-holomorphic functions $K^{(1)},\ldots,K^{(t)}:\Delta(0;r)\to\mathcal{M}$. The formula

$$q(D)(z^\alpha g)=\sum_{k\le\alpha}\binom{\alpha}{k}z^{\alpha-k}\frac{\partial^k q}{\partial z^k}(D)(g) \tag{2.4}$$

gives

$$q_i^*(\bar D)K(\cdot,w)|_{w=0}=\sum_{j=1}^t\{K^{(j)}(\cdot,w)|_{w=0}\}\{q_i^*(\bar D)\overline{g_j(w)}|_{w=0}\}$$

for $q_i\in\tilde{\mathbb{V}}_0(\mathcal{I})$, $1\le i\le t$. The proof follows from the fact that $\mathbb{V}_w(\mathcal{I})=\mathbb{V}_w(\mathcal{M})=\mathbb{V}_w(\mathcal{S}^{\mathcal{M}}_w)$.

Remark 2.3. We give details of the case where the ideal $\mathcal{I}$ is singly generated, namely $\mathcal{I}=<p>$. From [8], it follows that the reproducing kernel K admits a global factorization, that is, $K(z,w)=p(z)\chi(z,w)\bar p(w)$ for $z,w\in\Omega$ where $\chi(w,w)\neq0$ for all $w\in\Omega$. So we get $K_1(\cdot,w)=p(\cdot)\chi(\cdot,w)$ for all $w\in\Omega$. The proposition above gives a way to write down this section in term of reproducing kernel. Let $0\in V(\mathcal{I})$. Let q_0 be the lowest degree term in p. We claim that $[q_0^*]$ gives a non-trivial class in $\tilde{\mathbb{V}}_0(\mathcal{I})/\mathbb{V}_0(\mathcal{I})$. This is because all partial derivatives of q_0^* have degree less than that of q_0^* and hence from (2.4)

$$q_0^*(D)(z^\alpha g)|_0=\frac{\partial^\alpha q_0^*}{\partial z^\alpha}(D)(p)\big|_0=0 \text{ for all multi-indices } \alpha \text{ such that } |\alpha|>0$$

and thus $\frac{\partial q_0^*}{\partial z_i}\in\mathbb{V}_0(\mathcal{I})$ for all i, $1\le i\le m$, that is, $q_0^*\in\tilde{\mathbb{V}}_0(\mathcal{I})$. Also as the lowest degree of $p-q_0$ is strictly greater than that of q_0,

$$q_0^*(D)p|_0=q_0^*(D)(p-q_0+q_0)|_0=q_0^*(D)q_0|_0=\parallel q_0\parallel_0^2>0$$

This shows that $q_0^*\notin\mathbb{V}_0(\mathcal{I})$ and hence gives a non-trivial class in $\tilde{\mathbb{V}}_0(\mathcal{I})/\mathbb{V}_0(\mathcal{I})$. Therefore from the proof of Lemma 2.1, we have

$$q_0(\bar D)K(\cdot,w)|_{w=0}=K_1(\cdot,w)|_{w=0}q_0(\bar D)\overline{p(w)}|_0=\parallel q_0^*\parallel_0^2 K_1(\cdot,w)|_{w=0}.$$

Let q_{w_0} denotes the lowest degree term in $z - w_0$ in the expression of p around w_0. Then we can write

$$K_1(\cdot, w)|_{w=w_0} = \begin{cases} \frac{K(\cdot,w)|_{w=w_0}}{p(w_0)} & \text{if } w_0 \notin V(\mathcal{I}) \cap \Omega \\ \frac{q_{w_0}(\bar{D})K(\cdot,w)|_{w=w_0}}{\|q_{w_0}^*\|_{w_0}^2} & \text{if } w_0 \in V(\mathcal{I}) \cap \Omega. \end{cases} \tag{2.5}$$

For a fixed set of polynomials $q_1, \ldots, q_t$, the next lemma provides a sufficient condition for the classes $[q_1^*], \ldots, [q_t^*]$ to be linearly independent in $\tilde{\mathbb{V}}_{w_0}(\mathcal{I})/\mathbb{V}_{w_0}(\mathcal{I})$. The ideas involved in the two easy but different proofs given below will be used repeatedly in the sequel.

Lemma 2.4. *Let $q_1, \ldots, q_t$ are linearly independent polynomials in the polynomial ideal $\mathcal{I}$ such that $q_1, \ldots, q_t \in \tilde{\mathbb{V}}_0(\mathcal{I})$. Then $[q_1^*], \ldots, [q_t^*]$ are linearly independent in $\tilde{\mathbb{V}}_{w_0}(\mathcal{I})/\mathbb{V}_{w_0}(\mathcal{I})$.*

First Proof. Suppose $\sum_{i=1}^t \alpha_i [q_i^*] = 0$ in $\tilde{\mathbb{V}}_{w_0}(\mathcal{I})/\mathbb{V}_{w_0}(\mathcal{I})$ for some $\alpha_i \in \mathbb{C}$, $1 \le i \le t$. Thus $\sum_{i=1}^t \alpha_i q_i^* = q$ for some $q \in \mathbb{V}_{w_0}(\mathcal{I})$. Taking the inner product of $\sum_{i=1}^t \alpha_i q_i^*$ with q_j for a fixed j, we get

$$\sum_{i=1}^t \langle q_j, q_i \rangle_{w_0} = \big(\sum_{i=1}^t \alpha_i q_i^*\big)(D) q_j|_{w_0} = q(D) q_j|_{w_0} = 0.$$

The Grammian $\big(\!\big(\langle q_j, q_i \rangle_{w_0}\big)\!\big)_{i,j=1}^t$ of the linearly independent polynomials $q_1, \ldots, q_t$ is non-singular. Thus $\alpha_i = 0$, $1 \le i \le t$, completing the proof.
Second Proof. If $[q_1^*], \ldots, [q_t^*]$ are not linearly independent, then we may assume without loss of generality that $[q_1^*] = \sum_{i=2}^t \alpha_i [q_i^*]$ for $\alpha_1, \ldots, \alpha_t \in \mathbb{C}$. Therefore $[q_1^* - \sum_{i=2}^t \alpha_i p_i^*] = 0$ in the quotient space $\tilde{\mathbb{V}}_{w_0}(\mathcal{I})/\mathbb{V}_{w_0}(\mathcal{I})$, that is, $q_1^* - \sum_{i=2}^t \alpha_i q_i^* \in \mathbb{V}_{w_0}(\mathcal{I})$. So, we have

$$(q_1^* - \sum_{i=2}^t \alpha_i q_i^*)(D) q|_{w_0} = 0 \text{ for all } q \in \mathcal{I}.$$

Taking $q = q_1 - \sum_{i=2}^t \bar{\alpha}_i q_i$ we have $\| \ q_1 - \sum_{i=2}^t \bar{\alpha}_i q_i \ \|_{w_0}^2 = 0$. Hence $q_1 = \sum_{i=2}^t \bar{\alpha}_i q_i$ which is a contradiction. □

Suppose are $p_1, ..., p_t$ are a minimal set of generators for $\mathcal{I}$. Let $\mathcal{M}$ be the completion of $\mathcal{I}$ with respect to some inner product induced by a positive definite kernel. We recall from [9] that $\text{rank}_{\mathbb{C}}[\underline{z}]\mathcal{M} = t$. Let w_0 be a fixed but arbitrary point in Ω. We ask if there exist a choice of generators $q_1, ..., q_t$ such that $q_1^*(\bar{D})K(\cdot, w)_0, \ldots, q_t^*(\bar{D})K(\cdot, w)_0$ forms a basis for $\cap_{j=1}^m \ker(M_j - w_{0j})^*$. We isolates some instances where the answer is affirmative. However, this is not always possible (see remark 2.12). From [9, Lemma 5.11, Page-89], we have

$$\dim \cap_{j=1}^m \ker M_j* = \dim \mathcal{M}/\mathfrak{m}_0\mathcal{M} = \dim \mathcal{M} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_0 \le \text{rank}_{\mathbb{C}[\underline{z}]}\mathcal{M}.\dim \mathbb{C}_0 \le t,$$

where $\mathfrak{m}_0$ denotes the maximal ideal of $\mathbb{C}[\underline{z}]$ at 0. So we have $\dim \cap_{j=1}^m \ker M_j^* \le t$. The germs $p_{10}, \ldots, p_{t0}$ forms a set of generators, not necessarily minimal, for $\mathcal{S}_0^{\mathcal{M}}$. However minimality can be assured under some additional hypothesis. For example, let $\mathcal{I}$ be the ideal generated by the polynomials $z_1(1 + z_1), z_1(1 - z_2), z_2^2$. This is minimal set of generators for the ideal $\mathcal{I}$, hence for $\mathcal{M}$, but not for $\mathcal{S}_0^{\mathcal{M}}$. Since $\{z_1, z_2\}$ is a minimal set of generators for $\mathcal{S}_0^{\mathcal{M}}$, it follows that $\{z_1(1 + z_1), z_1(1 - z_2), z_2^2\}$ is not minimal for $\mathcal{S}_0^{\mathcal{M}}$. This was pointed out by R. G. Douglas.

Lemma 2.5. *Let $p_1, \ldots, p_t$ be homogeneous polynomials, not necessarily of the same degree. Let $\mathcal{I} \subset \mathbb{C}[\underline{z}]$ be an ideal for which $p_1, \ldots, p_t$ is a minimal set of generators. Let $\mathcal{M}$ be a submodule of an analytic Hilbert module over $\mathbb{C}[\underline{z}]$ such that $\mathcal{M} = [\mathcal{I}]$. Then the germs $p_{10}, \ldots, p_{t0}$ at 0 forms a minimal set of generators for $\mathcal{S}_0^{\mathcal{M}}$.*

Proof. For $1 \le i \le t$, let deg $p_i = \alpha_i$. Without loss of generality we assume that $\alpha_i \le \alpha_{i+1}$, $1 \le i \le t-1$. Suppose the germs $p_{10}, \ldots, p_{t0}$ are not minimal, that is, there exist $k (1 \le k \le t)$, $p_k = \sum_{i=1, i \ne k}^t \phi_i p_i$

for some choice of holomorphic functions ϕ_i, $1 \leq i \leq t, i \neq k$ defined on a suitable small neighborhood of 0. Thus we have

$$p_k = \sum_{i:\alpha_i \leq \alpha_k} \phi_i^{\alpha_k - \alpha_i} p_i,$$

where $\phi_i^{\alpha_k-\alpha_i}$ is the Taylor polynomial containing of ϕ_i of degree $\alpha_k - \alpha_i$. Therefore $p_1, \ldots, p_t$ can not be a minimal set of generators for the ideal $\mathcal{I}$. This contradiction completes the proof. $\square$

Consider the ideal $\mathcal{I}$ generated by the polynomials $z_1 + z_2 + z_1^2, z_2^3 - z_1^2$. We will see later that the joint kernel at 0, in this case is spanned by the independent vectors $p(\bar{D})K(\cdot, w)|_{w=0}, q(\bar{D})K(\cdot, w)|_{w=0}$, where $p = z_1 + z_2$ and $q = (z_1 - z_2)^2$. Therefore any vectors in the joint kernel is of the form $(\alpha p + \beta q)(\bar{D})K(\cdot, w)|_{w=0}$ for some $\alpha, \beta \in \mathbb{C}$. It then follows that $\alpha p + \beta q$ and $\alpha' p + \beta' q$ can not be a set of generators of $\mathcal{I}$ for any choice of $\alpha, \beta, \alpha', \beta' \in \mathbb{C}$. However in certain cases, this is possible. We describe below the case where $\{p_1(\bar{D})K(\cdot, w)|_{w=0}, ..., p_t(\bar{D})K(\cdot, w)|_{w=0}\}$ forms a basis for $\cap_{j=1}^n \ker M_j^*$ for an obvious choice of generating set in $\mathcal{I}$.

Lemma 2.6. *Let $p_1, \ldots, p_t$ be homogeneous polynomials of same degree. Suppose that $\{p_1, \ldots, p_t\}$ is a minimal set of generators for the ideal $\mathcal{I} \subset \mathbb{C}[\underline{z}]$. Then the set*

$$\{p_1(\bar{D})K(\cdot, w)|_{w=0}, ..., p_t(\bar{D})K(\cdot, w)|_{w=0}\}$$

forms a basis for $\cap_{j=1}^n \ker M_j^*$.

Proof. For $1 \leq i \leq t$, let deg $p_i = k$. It is enough to show, using Lemma 2.1, 2.4 and 2.5, that the polynomials $p_1^*, \ldots, p_t^*$ are in $\tilde{\mathbb{V}}_0(\mathcal{I})$. Since $\frac{\partial p_i^*}{\partial z_j}$ is of degree at most $k - 1$ for each i and j, $1 \leq i \leq t$, $1 \leq j \leq m$, and the the term of lowest degree in each polynomial in the ideal $p \in \mathcal{I}$ will be at least of degree k, it follows that $\frac{\partial p_i^*}{\partial z_j}(D)p|_0 = 0$, $p \in \mathcal{I}$, $1 \leq i \leq t, 1 \leq j \leq m$. This completes the proof. $\square$

Example 2.7. Let $\mathcal{M}$ be an analytic Hilbert module over $\Omega \subseteq \mathbb{C}^m$, and $\mathcal{M}_n$ be a submodule of $\mathcal{M}$ formed by the closure of polynomial ideal $\mathcal{I}$ in $\mathcal{M}$ where $\mathcal{I} = \langle z^\alpha = z_1^{\alpha_1} ... z_m^{\alpha_m} : \alpha_i \in \mathbb{N} \cup \{0\}, |\alpha| = \sum_{i=1}^m \alpha_i = n \rangle$. We note that $Z(\tau) = \{0\}$. Let K_n be the reproducing kernel corresponding to $\mathcal{M}_n$. Then,

(1) $\mathcal{M}_n = \{f \in \mathcal{M} : \partial^\alpha f(0) = 0, \text{ for } \alpha_i \in \mathbb{N} \cup \{0\}, |\alpha| \leq n - 1\}$

(2) $\bigcap_{j=1}^m \ker(M_j|_{\mathcal{M}_n} - w_j)^* = \begin{cases} \text{span}\{\mathrm{K_n}(\cdot, \mathrm{w})\}, & \text{for } w \neq 0; \\ \text{span}\{\bar{\partial}^\alpha \mathrm{K_n}(\cdot,\mathrm{w})|_{w=0} : \alpha_i \in \mathbb{N} \cup \{0\}, |\alpha| = n\}, & \text{for } w = 0. \end{cases}$

We now go further and show that a similar description of the joint kernel is possible even if the restrictive assumption of "same degree" is removed. We begin with the simple case of two generators.

Proposition 2.8. *Suppose $\{p_1, p_2\}$ is a minimal set of generators for the ideal $\mathcal{I}$. and are homogeneous with deg $p_1 \neq$ deg p_2. Let K be the reproducing kernel corresponding the Hilbert module $[\mathcal{I}]$, which is assumed to be in $\mathfrak{B}_1(\Omega^*)$. Then there exist polynomials q_1, q_2 which generate the ideal $\mathcal{I}$ and*

$$\{q_1(\bar{D})K(\cdot, w)|_{w=0},\ q_2(\bar{D})K(\cdot, w)|_{w=0}\}$$

is a basis for $\cap_{j=1}^m \ker M_j^*$.

Proof. Let deg $p_1 = k$ and deg $p_2 = k + n$ for some $n \geq 1$. The set $\{p_1, p_2 + (\sum_{|i|=n} \gamma_i z^i)p_1\}$ is a minimal set of generators for $\mathcal{I}$, $\gamma_i \in \mathbb{C}$ where $i = (i_1, \ldots, i_m)$ and $|i| = i_1 + \ldots + i_m$. We will take $q_1 = p_1$ and find constants γ_i in $\mathbb{C}$ such that

$$q_2 = p_2 + (\sum_{|i|=n} \gamma_i z^i)p_1.$$

We have to show (Lemma 2.1) that $\{[q_1^*], [q_2^*]\}$ is a basis in $\tilde{\mathbb{V}}_0(\mathcal{I})/\mathbb{V}_0(\mathcal{I})$. From the equation (2.2) and Lemma 2.4, it is enough to show that q_2^* is a in $\tilde{\mathbb{V}}_0(\mathcal{I})$. To ensure that $\frac{\partial q_2^*}{\partial z_k} \in \mathbb{V}_0(\mathcal{I})$, $1 \leq k \leq m$, we

need to check:

$$\frac{\partial^{|\alpha|}q_2^*}{\partial z^\alpha}(D)p_i|_{w=0} = \langle p_i, \frac{\partial^{|\alpha|}q_2}{\partial z^\alpha}\rangle|_0 = 0,$$

for all multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $1 \le |\alpha| \le n$ and $i = 1, 2$. For $|\alpha| > n$, these conditions are evident. Since the degree of the polynomial q_2 is $k+n$, we have $\langle p_2, \frac{\partial^{|\alpha|}q_2}{\partial z^\alpha}\rangle_0 = 0$, $1 \le |\alpha| \le n$. If $n > 1$, then $\langle p_1, \frac{\partial^{|\alpha|}q_2}{\partial z^\alpha}\rangle_0 = 0$, $1 \le |\alpha| < n$. To find γ_i, $i = (i_1, \ldots, i_m)$, we solve the equation $\langle p_1, \frac{\partial^{|\alpha|}q_2}{\partial z^\alpha}\rangle|_0 = 0$ for all α such that $|\alpha| = n$. By the Leibnitz rule,

$$\begin{aligned} \frac{\partial^{|\alpha|}q_2^*}{\partial z^\alpha} &= \frac{\partial^{|\alpha|}p_2^*}{\partial z^\alpha} + \sum_{\nu\le\alpha}\binom{\alpha}{\nu}\partial^{\alpha-\nu}(\sum_{|i|=n}\bar\gamma_i z^i)\frac{\partial^{|\nu|}p_1^*}{\partial z^\nu} \\ &= \frac{\partial^{|\alpha|}p_2^*}{\partial z^\alpha} + \sum_{\nu\le\alpha}\binom{\alpha}{\nu}(\sum_{|i|=n, i\ge\alpha-\nu}\bar\gamma_i\frac{i!}{(i-\alpha+\nu)!}z^{i-\alpha+\nu})\frac{\partial^{|\nu|}p_1^*}{\partial z^\nu}. \end{aligned}$$

Now $\frac{\partial^{|\alpha|}p^*}{\partial z^\alpha}(D)p_i|_{w=0} = 0$ gives

$$\begin{aligned} 0 &= \Big(\frac{\partial^{|\alpha|}p_2^*}{\partial z^\alpha} + \sum_{\nu\le\alpha}\binom{\alpha}{\nu}\Big(\sum_{|i|=n, i\ge\alpha-\nu}\bar\gamma_i\frac{i!}{(i-\alpha+\nu)!}z^{i-\alpha+\nu}\Big)\frac{\partial^{|\nu|}p_1^*}{\partial z^\nu}\Big)(D)p_1|_{w=0} \\ &= \langle p_1, \frac{\partial^{|\alpha|}p_2}{\partial z^\alpha}\rangle_0 + \sum_{r=0}^n\sum_{|i|=n}\overline{A_{\alpha i}(r)}\bar\gamma_i, \end{aligned} \tag{2.6}$$

where given the multi-indices α, i,

$$A_{\alpha i}(r) = \begin{cases} \sum_\nu \binom{\alpha}{\nu}\frac{i!}{(i-\alpha+\nu)!}\langle\frac{\partial^{|\nu|}p_1}{\partial z^\nu}, \frac{\partial^{|i-\alpha+\nu|}p_1}{\partial z^{i-\alpha+\nu}}\rangle_0 & |\nu| = r, \nu\le\alpha, i\ge\alpha-\nu; \\ 0 & \text{otherwise.}\end{cases} \tag{2.7}$$

Let $A(r) = \big(\!\big(A_{\alpha i}(r)\big)\!\big)$ be the $\binom{n+m-1}{m-1} \times \binom{n+m-1}{m-1}$ matrix in colexicographic order on α and i. Let $A = \sum_{r=0}^n A(r)$ and γ_n be the $\binom{n+m-1}{m-1} \times 1$ column vector $(\gamma_i)_{|i|=n}$. Thus the equation (2.6) is of the form

$$\bar A\bar\gamma_n = \Gamma, \tag{2.8}$$

where Γ is the $\binom{n+m-1}{m-1} \times 1$ column vector $(-\langle p_1, \frac{\partial^{|\alpha|}p_2}{\partial z^\alpha}\rangle_0)_{|\alpha|=n}$. Invertibility of the coefficient matrix A then guarantees the existence of a solution to the equation (2.8). We show that the matrix $A(r)$ is non-negative definite and the matrix $A(0)$ is diagonal:

$$A(0)_{\alpha i} = \begin{cases} \alpha! \parallel p_1 \parallel^2 & \text{if } \alpha = i \\ 0 & \text{if } \alpha \ne i. \end{cases} \tag{2.9}$$

and therefore positive definite. Fix a r, $1 \le r \le n$. To prove that $A(r)$ is non-negative definite, we show that it is the Grammian with respect to Fock inner product at 0. To each $\mu = (\mu_1, \ldots, \mu_m)$ such that $|\mu| = n - r$, we associate a $1 \times \binom{n+m-1}{m-1}$ tuple of polynomials X_μ^r, defined as follows

$$X_\mu^r(\beta) = \begin{cases} \mu!\binom{\beta}{\beta-\mu}\frac{\partial^{|\beta-\mu|}p_1}{\partial z^{\beta-\mu}} & \text{if } \beta \ge \mu \\ 0 & \text{otherwise,} \end{cases}$$

where $\beta = (\beta_1, \ldots, \beta_m)$, $|\beta| = n$ ($\beta \geq \mu$ if and only if $\beta_i \geq \mu_i$ for all i). By $X^r_\mu \cdot (X^r_\mu)^t$, we denote the $\binom{n+m-1}{m-1} \times \binom{n+m-1}{m-1}$ matrix whose αi-th element is $\langle X^r_\mu(\alpha), X^r_\mu(i)\rangle_0$, $|\alpha| = n = |i|$. We note that

$$
\begin{aligned}
(2.10) \sum_{|\mu|=n-r} \frac{1}{\mu!}(X^r_\mu \cdot (X^r_\mu)^t)_{\alpha i} &= \sum_{|\mu|=n-r} \frac{1}{\mu!}\langle X^r_\mu(\alpha), X^r_\mu(i)\rangle_0 \\
&= \sum_{|\mu|=n-r,\alpha\geq\mu,i\geq\mu} \frac{1}{\mu!}\langle \mu!\binom{\alpha}{\alpha-\mu}\frac{\partial^{|\alpha-\mu|}p_1}{\partial z^{\alpha-\mu}}, \mu!\binom{i}{i-\mu}\frac{\partial^{|i-\mu|}p_1}{\partial z^{i-\mu}}\rangle_0 \\
&= \sum_{|\nu|=r,\nu\leq\alpha,i\geq\alpha-\nu} (\alpha-\nu)!\binom{\alpha}{\nu}\binom{i}{i-\alpha+\nu}\langle\frac{\partial^{|\alpha-\mu|}p_1}{\partial z^{\alpha-\mu}}, \frac{\partial^{|i-\mu|}p_1}{\partial z^{i-\mu}}\rangle_0 \\
&= A_{\alpha i}(r).
\end{aligned}
$$

Since $X^r_\mu \cdot (X^r_\mu)^t$ is the Grammian of the vector tuple X^r_μ, it is non-negative definite. Hence $A(r) = \sum_{|\mu|=n-r} \frac{1}{\mu!}(X^r_\mu \cdot (X^r_\mu)^t)$ is non-negative definite. Therefore A is positive definite and hence equation (2.8) admits a solution, completing the proof. □

Let $\mathcal{I}$ be a homogeneous polynomial ideal. As one may expect, the proof in the general case is considerably more involved. However the idea of the proof is similar to the simple case of two generators. Let $p_1, \ldots, p_v$ be a minimal set of generators, consisting of homogeneous polynomials, for the ideal $\mathcal{I}$. We arrange the set $\{p_1, \ldots, p_v\}$ in blocks of polynomials $P^1, \ldots, P^k$ according to ascending order of their degree, that is,

$$\{P^1, \ldots, P^k\} = \{p^1_1, \ldots, p^1_{u_1}, p^2_1, \ldots, p^2_{u_2}, \ldots, p^l_1, \ldots, p^l_{u_l}, \ldots, p^k_1, \ldots, p^k_{u_k}\},$$

where each $P^l = \{p^l_1, \ldots, p^l_{u_l}\}$, $1 \leq l \leq k$ consists of homogeneous polynomials of the same degree, say n_l and $n_{l+1} > n_l$, $1 \leq l \leq k-1$. As before, for $l = 1$, we take $q^1_j = p^1_j$, $1 \leq j \leq u_1$ and for $l \geq 2$ take

$$q^l_j = p^l_j + \sum_{f=1}^{l-1}\sum_{s=1}^{u_f} \gamma^{fs}_{lj} p^f_s, \text{ where } \gamma^{fs}_{lj}(z) = \sum_{|i|=n_l-n_f} \gamma^{fs}_{lj}(i) z^i.$$

Each γ^{fs}_{lj} is a polynomial of degree $n_l - n_f$ for some choice of $\gamma^{fs}_{lj}(i)$ in $\mathbb{C}$. So we obtain another set of polynomials $\{Q^1, \ldots, Q^k\}$ with $Q^l = \{q^l_1, \ldots, q^l_{u_l}\}$, $1 \leq l \leq k$ satisfying the the same property as the set of polynomials $\{P^1, \ldots, P^k\}$. From Lemma 2.1 and 2.4, it is enough to check q^{l*}_j is in $\tilde{\mathbb{V}}_0(\mathcal{I})$. This condition yields a linear system of equation as in the proof of Proposition 2.8, except that the co-efficient matrix is a block matrix with each block similar to A defined by the equation (2.7). For q^{l*}_j in $\tilde{\mathbb{V}}_0(\mathcal{I})$, the constants $\gamma^{fs}_{lj}(i)$ must satisfy:

$$
\begin{aligned}
0 &= \frac{\partial^{|\alpha|} q^{l*}_j}{\partial z^\alpha}(D) p^e_t|_0 \\
&= \langle p^e_t, \frac{\partial^{|\alpha|} p^l_j}{\partial z^\alpha}\rangle_0 + \sum_{f=1}^{l-1}\sum_{s=1}^{u_f}\sum_{\nu\leq\alpha}\binom{\alpha}{\nu}\sum_{|i|=n_l-n_f,i\geq\alpha-\nu}\overline{\gamma^{fs}_{lj}(i)}\frac{i!}{(i-\alpha+\nu)!}\langle\frac{\partial^{|i-\alpha+\nu|}p^e_t}{\partial z^{i-\alpha+\nu}}, \frac{\partial^{|\nu|}p^f_s}{\partial z^\nu}\rangle_0
\end{aligned}
$$

All the terms in the equation are zero except when $|\alpha| = n_l - n_d$, $1 \leq d \leq l-1$. For $e = d = f$, we have the equations

$$(2.11) \qquad -\langle p^d_t, \frac{\partial^{|\alpha|}p^l_j}{\partial z^\alpha}\rangle_0 = \sum_{s=1}^{u_d}\sum_{r=0}^{n_l-n_d}\sum_{|i|=n_l-n_d}\overline{\left(A^d_{st}(r)\right)_{\alpha i}\gamma^{ds}_{lj}(i)},$$

where

$$\left(A^d_{st}(r)\right)_{\alpha i} = \begin{cases} \sum_\nu \binom{\alpha}{\nu} \frac{i!}{(i-\alpha+\nu)!} \langle \frac{\partial^{|\nu|} p^d_s}{\partial z^\nu}, \frac{\partial^{|i-\alpha+\nu|} p^d_t}{\partial z^{i-\alpha+\nu}} \rangle_0 & |\nu| = r, \nu \le \alpha, i \ge \alpha - \nu; \\ 0 & \text{otherwise.} \end{cases}$$

Let $A^d_{st}(r)$ be the $\binom{n_l - n_{d-1} + m-1}{m-1} \times \binom{n_l - n_{d-1} + m-1}{m-1}$ matrix whose αi-th element is $\left(A^d_{st}(r)\right)_{\alpha i}$. We consider the block-matrix $A^d(r) = (A^d_{st}(r))$, $1 \le s, t \le u_d$.

Fix a r, $1 \le r \le n_l - n_d$. To each $\mu = (\mu_1, \ldots, \mu_m)$ such that $|\mu| = n_l - n_d - r$, associate a $1 \times \binom{n_l - n_d + m - 1}{m-1}$ tuple of polynomials $X^{ds}_{\mu r}$ defined as follows:

$$X^{ds}_{\mu r}(\beta) = \begin{cases} \mu! \binom{\beta}{\beta - \mu} \frac{\partial^{|\beta - \mu|} p^d_s}{\partial z^{\beta - \mu}} & \text{if } \beta \ge \mu \\ 0 & \text{otherwise,} \end{cases}$$

where $\beta = (\beta_1, \ldots, \beta_m)$ with $|\beta| = n_l - n_d$. Let $X^d_{\mu r} = (X^{d1}_{\mu r}, \ldots, X^{d(n_l - n_d)}_{\mu r})$. Using same argument as in (2.9) and (2.10), we see that the matrix

$$A^d(r) = \sum_{|\mu| = n-r} \frac{1}{\mu!} (X^d_{\mu r} \cdot (X^d_{\mu r})^t)$$

is non-negative definite when $r \ge 0$ and $A^d(0)$ is positive definite. Thus $A^d = \sum_{r=0}^{n_l - n_d} A^d(r)$ is positive definite. Let

$$\gamma^d_{lj} = ((\gamma^{d1}_{lj}(i))_{|i| = n_l - n_d}, \ldots, (\gamma^{d(n_l - n_d)}_{lj}(i))_{|i| = n_l - n_d})^{tr},$$

where each $(\gamma^{ds}_{lj}(i))_{|i| = n_l - n_d}$ is a $\binom{n_l - n_d + m - 1}{m-1} \times 1$ column vector. Define

$$\Gamma^d_{lj} = ((-\langle p^d_1, \frac{\partial^{|\alpha|} p^l_j}{\partial z^\alpha} \rangle_0)_{|\alpha| = n_l - n_d}, \ldots, (-\langle p^d_{u_d}, \frac{\partial^{|\alpha|} p^l_j}{\partial z^\alpha} \rangle_0)_{|\alpha| = n_l - n_d}).$$

The equation (2.11) is then takes the form $\overline{A^d \gamma^d_{lj}} = \Gamma^d_{lj}$, which admits a solution (as A^d is invertible) for each d, l and j. Thus we have proved the following theorem.

Theorem 2.9. *Let $\mathcal{I} \subset \mathbb{C}[\underline{z}]$ be a homogeneous ideal and $\{p_1, \ldots, p_v\}$ be a minimal set of generators for $\mathcal{I}$ consisting of homogeneous polynomials. Let K be the reproducing kernel corresponding the Hilbert module $[\mathcal{I}]$, which is assumed to be in $\mathfrak{B}_1(\Omega^*)$. Then there exists a set of generators $q_1, ..., q_v$ for the ideal $\mathcal{I}$ such that the set $\{q_i(\bar{D}) K(\cdot, w)|_{w=0} : 1 \le i \le v\}$ is a basis for $\cap_{j=1}^n \ker M_j^*$.*

We remark that the new set of generators $q_1, \ldots, q_v$ for $\mathcal{I}$ is more or less "*canonical*"! It is uniquely determined modulo a linear transformation as shown below.

Let $\mathcal{I} \subset \mathbb{C}[\underline{z}]$ be an ideal. Suppose there are two sets of homogeneous polynomials $\{p_1, \ldots, p_v\}$ and $\{\tilde{p}_1, \ldots, \tilde{p}_v\}$ both of which are minimal set of generators for $\mathcal{I}$. Theorem 2.9 guarantees the existence of a new set of generators $\{q_1, \ldots, q_v\}$ and $\{\tilde{q}_1, \ldots, \tilde{q}_v\}$ corresponding to each of these generating sets with additional properties which ensures that the equality

$$[\tilde{q}_i^*] = \sum_{j=1}^v \alpha_{ij} [q_j^*], \ 1 \le i \le v$$

holds in $\tilde{\mathbb{V}}_0(\mathcal{I}) / \mathbb{V}_0(\mathcal{I})$ for some choice of complex constants α_{ij}, $1 \le i, j \le v$. Therefore $\tilde{q}_i^* - \sum_{i=1}^v \bar{\alpha}_{ij} q_j^* \in \mathbb{V}_0(\mathcal{I})$. Since $\tilde{q}_i - \sum_{i=1}^v \alpha_{ij} q_j$ is in $\mathcal{I}$, we have

$$0 = \left((\tilde{q}_i^* - \sum_{i=1}^v \bar{\alpha}_{ij} q_j^*)(D) \right) \left(\tilde{q}_i - \sum_{i=1}^v \alpha_{ij} q_j \right) = \| \tilde{q}_i - \sum_{i=1}^v \alpha_{ij} q_j \|_0^2, \ 1 \le i \le v,$$

and hence $\tilde{q}_i = \sum_{i=1}^v \alpha_{ij} q_j$, $1 \le i \le v$. We have therefore proved the following.

Proposition 2.10. *Let $\mathcal{I} \subset \mathbb{C}[\underline{z}]$ be a homogeneous ideal. If $\{q_1, \ldots, q_v\}$ is a minimal set of generators for $\mathcal{I}$ with the property that $\{[q_i^*] : 1 \le i \le v\}$ is a basis for $\tilde{\mathbb{V}}_0(\mathcal{I}) / \mathbb{V}_0(\mathcal{I})$, then $q_1, \ldots, q_v$ is unique up to a linear transformation.*

We end this section with the explicit calculation of the joint kernel for a class of submodules of the Hardy module which illustrate the methods of Proposition 2.8.

Example 2.11. Let p_1, p_2 be the minimal set of generators for an ideal $\mathcal{I} \subseteq \mathbb{C}[z_1, z_2]$. Assume that p_1, p_2 are homogeneous, $\deg p_2 = \deg p_1 + 1$ and $V(\mathcal{I}) = \{0\}$. As in Proposition 2.8, set $q_1 = p_1$ and $q_2 = p_2 + (\gamma_{10} z_1 + \gamma_{01} z_2) p_1$ subject to the equations

$$\begin{pmatrix} \| \partial_1 p_1 \|_0^2 + \| p_1 \|_0^2 & \langle \partial_2 p_1, \partial_1 p_1 \rangle_0 \\ \langle \partial_1 p_1, \partial_2 p_1 \rangle_0 & \| \partial_2 p_1 \|_0^2 + \| p_1 \|_0^2 \end{pmatrix} \begin{pmatrix} \gamma_{10} \\ \gamma_{01} \end{pmatrix} = - \begin{pmatrix} \langle p_1, \partial_1 p_2 \rangle_0 \\ \langle p_1, \partial_2 p_2 \rangle_0 \end{pmatrix} \tag{2.12}$$

In this special case, the invertibility of the coefficient matrix follows from the positivity (Cauchy - Schwarz inequality) of its determinant

$$\begin{aligned} &\| p_1 \|_0^4 + \| \partial_1 p_1 \|_0^2 \| p_1 \|_0^2 + \| \partial_2 p_1 \|_0^2 \| p_1 \|_0^2 \\ &\qquad\qquad + (\| \partial_1 p_1 \|_0^2 \| \partial_2 p_1 \|_0^2 - |\langle \partial_1 p_1, \partial_2 p_1 \rangle_0|^2). \end{aligned}$$

Specifically, if the ideal $\mathcal{I} \subset \mathbb{C}[z_1, z_2]$ is generated by $z_1 + z_2$ and z_2^2. We have $V(\mathcal{I}) = \{0\}$. The reproducing kernel K for $[\mathcal{I}] \subseteq H^2(\mathbb{D}^2)$ is

$$\begin{aligned} K_{[\mathcal{I}]}(z, w) &= \frac{1}{(1 - z_1 \bar{w}_1)(1 - z_2 \bar{w}_2)} - \frac{(z_1 - z_2)(\bar{w}_1 - \bar{w}_2)}{2} - 1 \\ &= \frac{(z_1 + z_2)(\bar{w}_1 + \bar{w}_2)}{2} + \sum_{i+j \geq 2}^{\infty} z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j. \end{aligned}$$

The vector $\bar{\partial}_2^2 K_{[\mathcal{I}]}(z, w)|_0 = 2 z_2^2$ is not in the joint kernel of $P_{[\mathcal{I}]}(M_1^*, M_2^*)|_{[\mathcal{I}]}$ since $M_2^*(z_2^2) = z_2$ and $P_{[\mathcal{I}]} z_2 = (z_1 + z_2)/2 \neq 0$. However, from the equation (2.12), we have $q_1 = z_1 + z_2$ and $q_2 = (z_1 - z_2)^2$, we see that q_1, q_2 generate the ideal $\mathcal{I}$ and $\{(\bar{\partial}_1 + \bar{\partial}_2) K(\cdot, w)|_0, (\bar{\partial}_1 - \bar{\partial}_2)^2 K(\cdot, w)|_0\}$ forms a basis of the joint kernel.

Remark on Example 2.11. Let $\widetilde{\mathcal{I}}$ be the ideal generated by z_1 and z_2^2. Since z_1 is not a linear combination of q_1 and q_2, it follows (Proposition 2.10) that $\mathcal{I} \neq \widetilde{\mathcal{I}}$. In fact Proposition 2.10 gives an effective tool to determine when a homogeneous ideal is monoidal. Let $\{q_1, \ldots, q_v\}$ be a canonical set of generators for $\mathcal{I}$. Let Λ be the collection of monomials in the expressions of $\{q_1, \ldots, q_v\}$. If the number of algebraically independent monomials in Λ is v, then $\mathcal{I}$ is monoidal.

Remark 2.12. If the generators of the ideal are not homogeneous then the conclusion of the theorem 2.9 is not valid. Take the ideal $\mathcal{I} \subset \mathbb{C}[z_1, z_2]$ generated by $z_1(1 + z_1), z_1(1 - z_2), z_2^2$ which is also minimal for $\mathcal{I}$. We have $V(\mathcal{I}) = \{0\}$. We note that the stalk $\mathcal{S}_0^{\mathcal{M}}$ at 0 is generated by z_1 and z_2^2. Similar calculations, as above, shows that $\{\bar{\partial}_1 K(\cdot, w)|_0, \bar{\partial}_2^2 K(\cdot, w)|_0\}$ is a basis of $\cap_{j=1}^2 \ker M_j^*$. But z_1 and z_2^2 can not be a set of generators for $\mathcal{I} \subset \mathbb{C}[z_1, z_2]$ which has rank 3. On the other hand, let $\mathcal{I}$ be the ideal generated by $z_1 + z_2 + z_1^2, z_2^3 - z_1^2$ which is minimal and $V(\mathcal{I}) = \{0\}$. In this case $\{(\bar{\partial}_1 + \bar{\partial}_2) K(\cdot, w)|_0, (\bar{\partial}_1 - \bar{\partial}_2)^2 K(\cdot, w)|_0\}$ is a basis of $\cap_{j=1,2} \ker M_j^*$. But $z_1 + z_2$ and $(z_1 - z_2)^2$ is not a generating set for the stalk at 0.

3. Resolution of Singularities

We will use the familiar technique of 'resolution of singularities' and construct the blow-up space of Ω along an ideal $\mathcal{I}$, which we will denote by $\hat{\Omega}$. There is a map $\pi : \hat{\Omega} \to \Omega$ which is biholomorphic on $\hat{\Omega} \setminus \pi^{-1}(V(\mathcal{I}))$. However, in general, $\hat{\Omega}$ need not even be a complex manifold. Abstractly, the inverse image sheaf of $\mathcal{S}^{\mathcal{M}}$ under π is locally principal and therefore corresponds to a line bundle on $\hat{\Omega}$. Here, we explicitly construct a holomorphic line bundle, via the monoidal transformation, on $\pi^{-1}(w_0)$, $w_0 \in V(\mathcal{I})$, and show that the equivalence class of these Hermitian holomorphic vector bundles are invariants for the Hilbert module $\mathcal{M}$.

In the paper [8], submodules of functions vanishing at the origin of $H^{(\lambda,\mu)}(\mathbb{D}^2)$ were studied using the blow-up $\mathbb{D}^2 \setminus (0,0) \cup \mathbb{P}^1$ of the bi-disc. This is also known as the quadratic transform. However, this technique yields useful information only if the generators of the submodule are homogeneous polynomials of same degree. The monoidal transform, as we will see below, has wider applicability.

For any two Hilbert module $\mathcal{M}_1$ and $\mathcal{M}_2$ in the class $\mathcal{B}_1(\Omega)$ and $L : \mathcal{M}_1 \to \mathcal{M}_2$ a module map between them, let $\mathcal{S}^L : \mathcal{S}^{\mathcal{M}_1}(V) \to \mathcal{S}^{\mathcal{M}_2}(V)$ be the map defined by

$$\mathcal{S}^L \sum_{i=1}^n f_i|_V g_i := \sum_{i=1}^n L f_i|_V g_i, \text{ for } f_i \in \mathcal{M}_1, g_i \in \mathcal{O}(V), n \in \mathbb{N}.$$

The map $\mathcal{S}^L$ is well defined: if $\sum_{i=1}^n f_i|_V g_i = \sum_{i=1}^n \tilde{f}_i|_V \tilde{g}_i$, then $\sum_{i=1}^n L f_i|_V g_i = \sum_{i=1}^n L\tilde{f}_i|_V \tilde{g}_i$. Suppose $\mathcal{M}_1$ is isomorphic to $\mathcal{M}_2$ via the unitary module map L. Now, it is easy to verify that $(\mathcal{S}^L)^{-1} = \mathcal{S}^{L^*}$. It then follows that $\mathcal{S}^{M_1}$ is isomorphic, as sheaves of modules over $\mathcal{O}_\Omega$, to $\mathcal{S}^{M_2}$ via the map $\mathcal{S}^L$.

Let K_i be the reproducing kernel corresponding to $\mathcal{M}_i$, $i = 1, 2$. We assume that the dimension of the zero sets $X_i = Z(\mathcal{M}_i)$ of the modules $\mathcal{M}_i$, $i = 1, 2$, is less or equal to $m - 2$. Recall that the stalk $\mathcal{S}_w^{\mathcal{M}_i}$ is $\mathcal{O}_w$ for $w \in \Omega \setminus X_1$, $i = 1, 2$. Let $X = X_1 \cup X_2$. From [2, Lemma 1.3] and [7, Theorem 3.7], it follows that there exists a non-vanishing holomorphic function $\phi : \Omega \setminus X \to \mathbb{C}$ such that $LK_1(\cdot, w) = \bar{\phi}(w)K_2(\cdot, w)$, $L^*f = \phi f$ and $K_1(z, w) = \phi(z)K_2(z, w)\bar{\phi}(w)$. The function $\psi = 1/\phi$ on $\Omega \setminus X$ (induced by the inverse of L, that is, L^*) is holomorphic. Since $\dim X \le m - 2$, by Hartog's theorem (cf. [14, Page 198]) there is a unique extension of ϕ to Ω such that ϕ is non-vanishing on Ω (ψ have an extension to Ω and $\phi\psi = 1$ on the open set $\Omega \setminus X$). Thus $X_1 = X_2$. For $w_0 \in X$, the stalks are not just isomorphic but equal:

$$\begin{aligned}
\mathcal{S}_{w_0}^{\mathcal{M}_1} &= \{\sum_{i=1}^n h_i g_i : g_i \in \mathcal{M}_1, h_i \in {}_m\mathcal{O}_{w_0}, 1 \le i \le n, n \in \mathbb{N}\} \\
&= \{\sum_{i=1}^n h_i \phi f_i : f_i \in \mathcal{M}_2, h_i \in {}_m\mathcal{O}_{w_0}, 1 \le i \le n, n \in \mathbb{N}\} \\
&= \{\sum_{i=1}^n \tilde{h}_i f_i : f_i \in \mathcal{M}_2, \tilde{h}_i \in {}_m\mathcal{O}_{w_0}, 1 \le i \le n, n \in \mathbb{N}\} = \mathcal{S}_{w_0}^{\mathcal{M}_2}.
\end{aligned}$$

The following theorem is modeled after the well known rigidity theorem which is obtained by taking $\mathcal{M} = \widetilde{\mathcal{M}}$. The proof below is different from the ones in [3] or [10] and uses the techniques developed in this paper and in [2]. We note the conditions in [10, Theorem 3.6] are same as the following theorem, as dimension of the algebraic variety $V(\mathcal{I})$ for some ideal $\mathcal{I} \subset \mathbb{C}[\underline{z}]$ is same as the holomorphic dimension by [15, Theorem 5.7.1].

Theorem 3.1. *Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be two Hilbert modules in $\mathfrak{B}_1(\Omega^*)$ consisting of holomorphic functions on a bounded domain $\Omega \subset \mathbb{C}^m$. Assume that the dimension of the zero set of these modules is at most $m - 2$. Suppose there exists polynomial ideals $\mathcal{I}$ and $\widetilde{\mathcal{I}}$ such that $\mathcal{M} = [\mathcal{I}]_{\mathcal{M}}$ and $\widetilde{\mathcal{M}} = [\widetilde{\mathcal{I}}]_{\widetilde{\mathcal{M}}}$. Assume that every algebraic component of $V(\mathcal{I})$ and $V(\widetilde{\mathcal{I}})$ intersects Ω. Then $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are equivalent if and only if $\mathcal{I} = \widetilde{\mathcal{I}}$.*

Proof. For $w_0 \in \Omega$, we have $\mathbb{V}_{w_0}(\mathcal{I}) = \mathbb{V}_{w_0}(\mathcal{S}_{w_0}^{\mathcal{M}})$ from [2, Lemma 3.2 and 3.3], and $\mathcal{S}_{w_0}^{\mathcal{M}} = \mathcal{S}_{w_0}^{\widetilde{\mathcal{M}}}$. Therefore $\mathbb{V}_{w_0}(\mathcal{I}) = \mathbb{V}_{w_0}(\widetilde{\mathcal{I}})$. In other words, setting $\mathcal{I}_{w_0}^e = \{p \in \mathbb{C}[\underline{z}] : q(D)p|_{w_0} = 0 \text{ for all } q \in \mathbb{V}_{w_0}(\mathcal{I})(= \mathbb{V}_{w_0}\}$, as in [3], we see that $\mathcal{I}_{w_0}^e = \widetilde{\mathcal{I}}_{w_0}^e$ for all $w_0 \in \Omega$. The proof is now complete since $\mathcal{I} = \cap_{w_0 \in \Omega} \mathcal{I}_{w_0}^e$ (cf. [3, Corollary 2.1.2]). $\square$

Example 3.2. For $j = 1, 2$, let $\mathcal{I}_j \subset \mathbb{C}[z_1, \ldots, z_m]$, $m > 2$, be the ideals generated by z_1^n and $z_1^{k_j} z_2^{n-k_j}$. Let $[\mathcal{I}_j]$ be the submodule in the Hardy module $H^2(\mathbb{D}^m)$. Now, from the Theorem proved above, it

follows that $[\mathcal{I}_1]$ is equivalent to $[\mathcal{I}_2]$ if and only if $\mathcal{I}_1 = \mathcal{I}_2$. We conclude, using Proposition 2.10, that these two ideals are same only if $k_1 = k_2$.

3.1. The Monoidal Transformation. Let $\mathcal{M}$ be a Hilbert module in $\mathfrak{B}_1(\Omega^*)$, which is the closure, in $\mathcal{M}$, of some polynomial ideal $\mathcal{I}$. Let K denote the corresponding reproducing kernel. Let $w_0 \in Z(\mathcal{M})$. Set

$$t = \dim \mathcal{S}^{\mathcal{M}}_{w_0}/\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0 0} = \dim \cap_{j=1}^m \ker(M_j - w_{0j})^* = \dim \tilde{\mathbb{V}}_{w_0}(\mathcal{I})/\mathbb{V}_{w_0}(\mathcal{I}).$$

By the decomposition theorem [2, Theorem 1.5], there exists a minimal set of generators $g_1, \cdots, g_t$ of $\mathcal{S}^{\mathcal{M}_1}_0$ and a $r > 0$ such that

$$K(\cdot, w) = \sum_{i=1}^t \overline{g_j(w)} K^{(j)}(\cdot, w) \text{ for all } w \in \Delta(w_0; r) \tag{3.1}$$

for some choice of anti-holomorphic functions $K^{(1)}, \ldots, K^{(t)} : \Delta(w_0; r) \to \mathcal{M}$.

Assume that $Z := Z(g_1, \ldots, g_t) \cap \Omega$ be a singularity free analytic subset of $\mathbb{C}^m$ of codimension t. We point out that Z depends on $\mathcal{M}$ as well as w_0. Define

$$\widehat{\Delta}(w_0; r) := \{(w, \pi(u)) \in \Delta(w_0; r) \times \mathbb{P}^{t-1} : u_i g_j(w) - u_j g_i(w) = 0,\ 1 \le i, j \le t\}.$$

Here the map $\pi : \mathbb{C}^t \setminus \{\underline{0}\} \to \mathbb{P}^{t-1}$ is given by $\pi(u) = (u_1 : \ldots : u_t)$, the corresponding projective coordinate. The space $\widehat{\Delta}(w_0; r)$ is the monoidal transformation with center Z ([12, page 241]). Consider the map $p := \mathrm{pr}_1 : \widehat{\Delta}(w_0; r) \to \Delta(w_0; r)$ given by $(w, \pi(z)) \mapsto w$. For $w \in Z$, we have $p^{-1}(w) = \{w\} \times \mathbb{P}^{t-1}$. This map is holomorphic and proper. Actually $p : \widehat{\Delta}(w_0; r) \setminus p^{-1}(Z) \to \Delta(w_0; r) \setminus Z$ is biholomorphic with $p^{-1} : w \mapsto (w, (g_1(w) : \ldots : g_t(w)))$. The set $E(\mathcal{M}) := p^{-1}(Z)$ which is $Z \times \mathbb{P}^{t-1}$, is called the exceptional set.

We describe a natural line bundle on the blow-up space $\widehat{\Delta}(w_0; r)$. Consider the open set $U_1 = (\Delta(w_0; r) \times \{u_1 \neq 0\}) \cap \widehat{\Delta}(w_0; r)$. Let $\frac{u_j}{u_1} = \theta^1_j$, $2 \le j \le t$. On this chart $g_j(w) = \theta^1_j g_j(w)$. From the decomposition given in the equation (3.1), we have

$$K(\cdot, w) = \overline{g_1(w)} \{K^{(1)}(\cdot, w) + \sum_{j=2}^t \bar{\theta}^1_j K^{(j)}(\cdot, w)\}.$$

This decomposition then yields a section on the chart U_1, of the line bundle on the blow-up space $\widehat{\Delta}(w_0; r)$:

$$s_1(w, \theta) = K^{(1)}(\cdot, w) + \sum_{j=2}^t \bar{\theta}^1_j K^{(j)}(\cdot, w).$$

The vectors $K^{(j)}(\cdot, w)$ are not uniquely determined. However, there exists a canonical choice of these vectors starting from a basis, $\{v_1, \ldots, v_t\}$, of the joint kernel $\cap_{i=1}^n \ker(M_j - w_j)^*$:

$$K(\cdot, w) = \sum_{j=1}^t \overline{g_j(w)} P(\bar{w}, \bar{w}_0) v_j,\ w \in \Delta(w_0; r)$$

for some $r > 0$ and generators $g_1, \ldots, g_t$ of the stalk $\mathcal{S}^{\mathcal{M}}_{w_0}$. Thus we obtain the canonical choice $K^{(j)}(\cdot, w) = P(\bar{w}, \bar{w}_0) v_j$, $1 \le j \le t$ (cf. [2, Section 6]). Let $\mathcal{L}(\mathcal{M})$ be the line bundle on the blow-up space $\widehat{\Delta}(w_0; r)$ determined by the section $(w, \theta) \mapsto s_1(w, \theta)$, where

$$s_1(w, \theta) = P(\bar{w}, \bar{w}_0) v_1 + \sum_{j=2}^t \bar{\theta}^1_j P(\bar{w}, \bar{w}_0) v_j,\ (w, \theta) \in U_1.$$

Let $\widetilde{\mathcal{M}}$ be a second Hilbert module in $\mathfrak{B}_1(\Omega^*)$, which is the closure of the polynomial ideal $\mathcal{I}$ with respect to another inner product. Assume that $\widetilde{\mathcal{M}}$ is equivalent to $\mathcal{M}$ via a unitary module map L. In

the proof of Theorem 1.10 in [2], we have shown that $LP(\bar{w},\bar{w}_0)=\widetilde{P}(\bar{w},\bar{w}_0)L$. Thus

$$\overline{\phi(w)}\widetilde{K}(\cdot,w)=LK(\cdot,w)=\sum_{j=1}^{t}\overline{g_j(w)}LP(\bar{w},\bar{w}_0)v_j=\sum_{j=1}^{t}\overline{g_j(w)}\widetilde{P}(\bar{w},\bar{w}_0)Lv_j.$$

Therefore $s_1(w,\theta)=\frac{1}{\overline{\phi(w)}}(\widetilde{P}(\bar{w},\bar{w}_0)Lv_1+\sum_{j=2}^{t}\bar{\theta}_j^1\widetilde{P}(\bar{w},\bar{w}_0)Lv_j)$ and

$$Ls_1(w,\theta)=\overline{\phi(w)}\tilde{s}_1(w,\theta).$$

Hence the line bundles $\mathcal{L}(\mathcal{M})$ and $\mathcal{L}(\widetilde{\mathcal{M}})$ are equivalent as Hermitian holomorphic line bundle on $\widehat{\Delta}(w_0;r)^*=\{(\bar{w},\pi(\bar{u})):(w,\pi(u))\in\widehat{\Delta}(w_0;r)\}$. Since $K^{(j)}(\cdot,w),1\leq j\leq t$ are linearly independent [2, Theorem 1.5], it follows that $Z(\mathcal{M})\cap\Delta(w_0;r)=Z$. Thus if $w\in\Delta(w_0;r)\setminus Z$, then $g_i(w)\neq 0$ for some i, $1\leq i\leq t$. Hence $s_i(w,\theta)=\frac{k(\cdot,w)}{g_i(w)}$ on $(\Delta(w_0;r)\times\{u_i\neq 0\})\cap\widehat{\Delta}(w_0;r)$. Therefore the restriction of the bundle $\mathcal{L}(\mathcal{M})$ to $\widehat{\Delta}(w_0;r)\setminus p^{-1}(Z)$ is the pull back of the Cowen-Douglas bundle for $\mathcal{M}$ on $\Delta(w_0;r)\setminus Z$, via the biholomorphic map π on $\widehat{\Delta}(w_0;r)\setminus p^{-1}(Z)$. we have therefore proved the following Theorem.

Theorem 3.3. *Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be two Hilbert modules in $\mathfrak{B}_1(\Omega)$ consisting of holomorphic functions on a bounded domain $\Omega\subset\mathbb{C}^m$. Assume that the dimension of the zero set of these modules is at most $m-2$. Suppose there exists a polynomial ideal $\mathcal{I}$ such that $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are the completions of $\mathcal{I}$ with respect to different inner product. Then $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are equivalent if and only if the line bundles $\mathcal{L}(\mathcal{M})$ and $\mathcal{L}(\widetilde{\mathcal{M}})$ are equivalent as Hermitian holomorphic line bundle on $\widehat{\Delta}(w_0;r)^*$.*

Although in general, Z need not be a complex manifold, The restriction of s_1 to $p^{-1}(w_0)$ for $w_0\in Z$ determines a holomorphic line bundle on $p^{-1}(w_0)^*:=\{(w_0,\pi(\bar{u})):(\bar{w}_0,\pi(u))\in p^{-1}(w_0)\}$, which we denote by $\mathcal{L}_0(\mathcal{M})$. Thus $s_1=s_1(w,\theta)|_{\{w_0\}\times\{u_i\neq 0\}}$ is given by the formula

$$s_1(\theta)=K^{(1)}(\cdot,w_0)+\sum_{j=2}^{t}\bar{\theta}_j^1K^{(j)}(\cdot,w_0).$$

Since the vectors $K^{(j)}(\cdot,w_0),1\leq j\leq t$ are uniquely determined by the generators $g_1,\ldots,g_t$, s_1 is well defined.

Theorem 3.4. *Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be two Hilbert modules in $\mathfrak{B}_1(\Omega)$ consisting of holomorphic functions on a bounded domain $\Omega\subset\mathbb{C}^m$. Assume that the dimension of the zero set of these modules is at most $\leq m-2$. Suppose there exists a polynomial ideal $\mathcal{I}$ such that $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are the completions of $\mathcal{I}$ with respect to different inner product. If the modules $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are equivalent, then the corresponding bundles $\mathcal{L}_0(\mathcal{M})$ and $\mathcal{L}_0(\widetilde{\mathcal{M}})$ they determine on the projective space $p^{-1}(w_0)^*$ for $w_0\in Z$, are equivalent as Hermitian holomorphic line bundle.*

Proof. Let $L:\mathcal{M}\to\widetilde{\mathcal{M}}$ be the unitary module map and K and $\widetilde{K}$ be the reproducing kernels corresponding to $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ respectively. The existence of a holomorphic function ϕ on $\Omega\setminus Z(\mathcal{M})$ such that $LK(\cdot,w)=\overline{\phi(w)}\widetilde{K}(\cdot,w)$, $L^*f=\phi f$ and $K(z,w)=\phi(z)\widetilde{K}(z,w)\overline{\phi(w)}$ follows from [2, Lemma 1.3] and [7, Theorem 3.7]. As we have pointed earlier, ϕ extends to a non-vanishing holomorphic function on Ω.

Since $\mathcal{M}$ is in $\mathfrak{B}_1(\Omega^*)$, it admits a decomposition as given in equation (3.1), with respect the generators $\tilde{g}_1,\ldots,\tilde{g}_t$ of $\mathcal{S}_{w_0}^{\widetilde{\mathcal{M}}}$. However, we may assume that $\tilde{g}_i=g_i$ for $1\leq i\leq t$, because $\mathcal{S}_{w_0}^{\mathcal{M}}=\mathcal{S}_{w_0}^{\widetilde{\mathcal{M}}}$ for all $w_0\in\Omega$. Thus

$$\widetilde{K}(\cdot,w)=\sum_{i=1}^{t}\overline{g_j(w)}\widetilde{K}^{(j)}(\cdot,w)\text{ for all }w\in\Delta(w_0;r)$$

For some $r > 0$. By applying the unitary L to equation (3.1), we get

$$\overline{\phi(w)}\widetilde{K}(\cdot, w) = LK(\cdot, w) \;=\; \sum_{i=1}^{t} \overline{g_j(w)} LK^{(j)}(\cdot, w).$$

Since ϕ does not vanish on Ω, we may choose

$$\widetilde{K}^{(j)}(\cdot, w) = \frac{LK^{(j)}(\cdot, w)}{\overline{\phi(w)}},\ 1 \leq j \leq t,\ w \in \Delta(w_0; r).$$

From part (iii) of the decomposition theorem ([2, Theorem 1.5]), the vectors $\widetilde{K}^{(j)}(\cdot, w_0)$, $1 \leq j \leq t$ are uniquely determined by the generators $g_1, \ldots, g_t$. Therefore $\widetilde{K}^{(j)}(\cdot, w_0) = \frac{LK^{(j)}(\cdot, w_0)}{\overline{\phi(w_0)}}$. Now the decomposition for $\widetilde{K}$ yields a holomorphic section $\tilde{s}_1(\theta) = \widetilde{K}^{(1)}(\cdot, w_0) + \sum_{j=2}^{t} \theta_j^1 \widetilde{K}^{(j)}(\cdot, w_0)$ for the holomorphic line bundle $\mathcal{L}_0(\widetilde{\mathcal{M}})$ on the projective space $p^{-1}(w_0)^*$. Therefore

$$\begin{aligned} Ls_1(\theta) \;&=\; LK^{(1)}(\cdot, w_0) + \sum_{j=2}^{t} \bar{\theta}_j^1 LK^{(j)}(\cdot, w_0) \\ &=\; \overline{\phi(w_0)}\{\widetilde{K}^{(1)}(\cdot, w_0) + \sum_{j=2}^{t} \bar{\theta}_j^1 \widetilde{K}^{(j)}(\cdot, w_0)\} \;=\; \overline{\phi(w_0)}\tilde{s}_1(\theta). \end{aligned}$$

From the unitarity of L, it follows that

$$\| s_1(\theta) \|^2 = \| Ls_1(\theta) \|^2 = |\phi(w_0)|^2 \| \tilde{s}_1(\theta) \|^2 \tag{3.2}$$

and consequently the Hermitian holomorphic line bundles $\mathcal{L}_0(\mathcal{M})$ and $\mathcal{L}_0(\widetilde{\mathcal{M}})$ on the projective space $p^{-1}(w_0)^*$ are equivalent. □

The existence of the polynomials $q_1, ..., q_t$ such that $K^{(j)}(\cdot, w)|_{w=w_0} = q_j^*(\bar{D})K(\cdot, w)|_{w=w_0}$, $1 \leq j \leq t$, is guaranteed by Lemma 2.1. The following Lemma shows that

$$\widetilde{K}^{(j)}(\cdot, w)|_{w=w_0} = q_j^*(\bar{D})\widetilde{K}(\cdot, w)|_{w=w_0},\ 1 \leq j \leq t$$

which makes it possible to calculate the section for the line bundles $\mathcal{L}_0(\mathcal{M})$ and $\mathcal{L}_0(\widetilde{\mathcal{M}})$ without any explicit reference to the generators of the stalks at w_0.

Lemma 3.5. *Let $\mathcal{I}$ be a polynomial ideal with dim $V(\mathcal{I}) \leq m - 2$ and K be the reproducing kernel of $[\mathcal{I}]$ which is assumed to be in $\mathfrak{B}_1(\Omega^*)$. Let $q_1, ..., q_t$ be the polynomials such that $K^{(j)}(\cdot, w)|_{w=w_0} = q_j^*(\bar{D})K(\cdot, w)|_{w=w_0}$. Let $\widetilde{K}$ be a reproducing kernel of $[\mathcal{I}]$, completed with respect to another inner product. Then $\widetilde{K}^{(j)}(\cdot, w)|_{w=w_0} = q_j^*(\bar{D})\widetilde{K}(\cdot, w)|_{w=w_0}$.*

Proof. For $f \in \mathcal{M}$ and $1 \leq i \leq m$, we have $\langle f, \bar{\partial}_i LK(\cdot, w)\rangle = \partial_i\langle f, LK(\cdot, w)\rangle = \partial_i\langle L^*f, K(\cdot, w)\rangle = \langle L^*f, \bar{\partial}_i K(\cdot, w)\rangle = \langle f, L\bar{\partial}_i K(\cdot, w)\rangle$, that is, $\bar{\partial}_i LK(\cdot, w) = L\bar{\partial}_i K(\cdot, w)$. Thus

$$p(\bar{D})LK(\cdot, w) = Lp(\bar{D})K(\cdot, w) \text{ for any } p \in \mathbb{C}[\underline{z}].$$

From equation (2.4), it follows that

$$\begin{aligned} LK^{(j)}(\cdot, w_0) \;&=\; L\{q_j^*(\bar{D})K(\cdot, w)|_{w=w_0}\} = \{Lq_j^*(\bar{D})K(\cdot, w)\}|_{w=w_0} \\ &=\; \{q_j^*(\bar{D})LK(\cdot, w)\}|_{w=w_0} \;=\; \{q_j^*(\bar{D})\overline{\phi(w)}\widetilde{K}(\cdot, w)\}|_{w=w_0} \\ &=\; [\sum_\alpha \bar{a}_\alpha\{q_j^*(\bar{D})(\bar{w} - \bar{w}_0)^\alpha \widetilde{K}(\cdot, w)\}]|_{w=w_0} \\ &=\; \sum_\alpha \bar{a}_\alpha \frac{\partial^\alpha q_j^*}{\partial z^\alpha}(\bar{D})\widetilde{K}(\cdot, w)|_{w=w_0}, \end{aligned}$$

where $\phi(w) = \sum_\alpha a_\alpha (w - w_0)^\alpha$, the power series expansion of ϕ around w_0. Now for any $p \in \mathcal{I}$ we have

$$\begin{aligned} \langle p, \frac{\partial^\alpha q_j^*}{\partial z^\alpha}(\bar{D})\widetilde{K}(\cdot, w)|_{w=w_0}\rangle &= \langle p, \frac{\partial^\alpha q_j^*}{\partial z^\alpha}(\bar{D})\widetilde{K}(\cdot, w)\rangle|_{w=w_0} \\ &= \frac{\partial^\alpha q_j}{\partial z^\alpha}(D)p(w)|_{w=w_0}. \end{aligned}$$

Since Lemma 2.1 ensures that $\{[q_1], \ldots, [q_t]\}$ is a basis for $\tilde{\mathbb{V}}_{w_0}(\mathcal{I})/\mathbb{V}_{w_0}(\mathcal{I})$, it follows that

$$\langle p, \frac{\partial^\alpha q_j^*}{\partial z^\alpha}(\bar{D})\widetilde{K}(\cdot, w)|_{w=w_0}\rangle = 0 \text{ for all } p \in \mathcal{I} \text{ and } \alpha > 0.$$

Therefore, we have $\frac{\partial^\alpha q_j^*}{\partial z^\alpha}(\bar{D})\widetilde{K}(\cdot, w)|_{w=w_0} = 0$ for $\alpha > 0$. Hence $LK^{(j)}(\cdot, w_0) = \bar{a}_0 q_j^*(\bar{D})\widetilde{K}(\cdot, w)|_{w=w_0} = \overline{\phi(w_0)} q_j^*(\bar{D})\widetilde{K}(\cdot, w)|_{w=w_0}$ and consequently $\widetilde{K}^{(j)}(\cdot, w)|_{w=w_0} = q_j^*(\bar{D})\widetilde{K}(\cdot, w)|_{w=w_0}$, $1 \le j \le t$. □

Remark 3.6. Let $\mathcal{M}$ Be a Hilbert module in $\mathfrak{B}_1(\Omega)$. Assume that $\mathcal{M} = [\mathcal{I}]_{\mathcal{M}}$ for some polynomial ideal $\mathcal{I}$ and the dimension of the zero set of $\mathcal{M}$ is $m - 1$. Let the polynomials $p_1, \ldots, p_t$ be a minimal set of generators for $\mathcal{M}$. Let $q = \text{g.c.d}\{p_1, \ldots, p_t\}$. Then the Beurling form (cf. [3]) of $\mathcal{I}$ is $q\mathcal{J}$, where $\mathcal{J}$ is generated by $\{p_1/q, \ldots, p_t/q\}$. From [3, Corollary 3.1.12], dim $V(\mathcal{J}) \le m - 2$ unless $\mathcal{J} = \mathbb{C}[\underline{z}]$. The reproducing kernels K of $\mathcal{M}$ is of the form $K(z, w) = q(z)\chi(z, w)\overline{q(w)}$. Let $\mathcal{M}_1$ be the Hilbert module determined by the non-negative definite kernel χ. The Hilbert module $\mathcal{M}$ is equivalent to $\mathcal{M}_1$. Now $\mathcal{M}_1 = [\mathcal{J}]$ and $Z(\mathcal{M}_1) = V(\mathcal{J})$. If $V(\mathcal{J}) = \phi$, then the modules $\mathcal{M}_1$ belongs to Cowen-Douglas class of rank 1. Otherwise, dim $V(\mathcal{J}) \le m - 2$ and Theorem 3.3 determines its equivalence class.

4. Examples

We illustrate, by means of some examples, the nature of the invariants we obtain from the line bundle $\mathcal{L}_0$ that lives on the projective space. From Theorem 3.4, it follows that the curvature of the line bundle $\mathcal{L}_0$ is an invariant for the submodule. An example was given in [8] showing that the curvature is is not a complete invariant. However the following lemma is useful for obtaining complete invariant in a large class of examples.

Lemma 4.1. *Let $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are Hilbert modules in $\mathfrak{B}_1(\Omega)$ for some bounded domain Ω in $\mathbb{C}^m$. Suppose that $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are such that they are in the Cowen-Douglas class $B_1(\Omega \setminus X)$ where $\dim X \le m - 2$. Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be any submodules of $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ respectively, such that*

- (i) $\mathbb{V}_w(\mathcal{M}) = \mathbb{V}_w(\widetilde{\mathcal{M}})$ *for all* $w \in \Omega$ *and*
- (ii) $\mathcal{M} = \cap_{w\in\Omega}\mathcal{M}_w^e$ *and* $\widetilde{\mathcal{M}} = \cap_{w\in\Omega}\widetilde{\mathcal{M}}_w^e$, *where as before* $\mathcal{M}_w^e := \{f \in \mathcal{H} : q(D)f|_w = 0$ *for all* $q \in \mathbb{V}_w(\mathcal{M})\}$.

If $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are equivalent, then $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are equivalent.

Proof. Suppose $U : \mathcal{H} \to \widetilde{\mathcal{H}}$ is a unitary module map. Then U is is a multiplication operator induced by some holomorphic function, say ψ, on $\Omega \setminus X$ (cf. [7]). This function ψ extends non-vanishingly to all of Ω by Hartog's Theorem. Let $w_0 \in \Omega$ and $q \in \mathbb{V}_{w_0}(\mathcal{M}) = \mathbb{V}_{w_0}(\widetilde{\mathcal{M}})$. Also let $\psi(w) = \sum_\alpha a_\alpha (w - w_0)^\alpha$ be the power series expansion around w_0. For $f \in \mathcal{M}$, we have

$$\begin{aligned} q(D)(Uf)|_{w=w_0} &= q(D)(\psi f)|_{w=w_0} = q(D)\{\sum_\alpha a_\alpha (w - w_0)^\alpha f\}|_{w=w_0} \\ &= \sum_\alpha a_\alpha q(D)\{(w - w_0)^\alpha f\}]|_{w=w_0} \\ &= \{\sum_{k\le\alpha} \binom{\alpha}{k} (w - w_0)^{\alpha-k} \frac{\partial^k q}{\partial z^k}(D)(f)\}_{w=w_0} \\ &= 0 \end{aligned}$$

since $\frac{\partial^k q}{\partial z^k} \in \mathbb{V}_{w_0}(\mathcal{M})$ for any multi index k whenever $q \in \mathbb{V}_{w_0}(\mathcal{M})$. Therefore it follows that $Uf \in \widetilde{\mathcal{M}}$. A similar arguments shows that $U^*\widetilde{\mathcal{M}} \subseteq \mathcal{M}$. The result follows from unitarity of U. $\square$

4.1. The (α, β, θ) examples: Weighted Bergman Modules in the unit ball. Let $\mathbb{B}^2 = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$ be the unit ball in $\mathbb{C}^2$. Let $L^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ be the Hilbert space of all (equivalence classes of) Borel measurable functions on $\mathbb{B}^2$ satisfying

$$\| f \|^2_{\alpha,\beta,\theta} = \int_{\mathbb{B}^2} |f(z)|^2 d\mu(z_1, z_2) < +\infty,$$

where the measure is

$$d\mu(z_1, z_2) = (\alpha + \beta + \theta + 2)|z_2|^{2\theta}(1 - |z_1|^2 - |z_2|^2)^\alpha(1 - |z_2|^2)^\beta dA(z_1, z_2)$$

for $(z_1, z_2) \in \mathbb{B}^2$, $-1 < \alpha, \beta, \theta < +\infty$ and $dA(z_1, z_2) = dA(z_1)dA(z_2)$. Here dA denote the normalized area measure in the plane, that is $dA(z) = \frac{1}{\pi}dxdy$ for $z = x + iy$. The weighted Bergman space $\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ is the subspace of $L^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ consisting of the holomorphic functions on $\mathbb{B}^2$. The Hilbert space $\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ is non-trivial if we assume that the parameters α, β, θ satisfy the additional condition:

$$\alpha + \beta + \theta + 2 > 0.$$

The reproducing kernel $K_{\alpha,\beta,\theta}$ of $\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ is given by

$$\begin{aligned} K_{\alpha,\beta,\theta}(z, w) &= \frac{1}{\alpha + \beta + \theta + 2} \frac{1}{(1 - z_1\bar{w}_1)^{\alpha+\beta+\theta+3}} \\ &\quad \times \left\{ \sum_{k=0}^{+\infty} \frac{(\alpha + \beta + \theta + k + 2)(\alpha + \theta + 2)_k}{(\theta + 1)_k} \left(\frac{z_2\bar{w}_2}{1 - z_1\bar{w}_1} \right)^k \right\}, \end{aligned}$$

where $z = (z_1, z_2), w = (w_1, w_2) \in \mathbb{B}^2$ and $(a)_k = a(a + 1) \ldots (a + k - 1)$ is the Pochhammer symbol. This kernel differs from the kernel $P_{\alpha,\beta,\theta}$ given in [13] only by a multiplicative constant. The reader may consult [13] for a detailed discussion of these Hilbert modules.

Let $\mathcal{I}_P$ be an ideal in $\mathbb{C}[z_1, z_2]$ such that $V(\mathcal{I}_P) = \{P\} \subset \mathbb{B}^2$. We have

$$\dim \ker D_{(M-w)^*} = \begin{cases} 1 & \text{for } w \in \mathbb{B}^2 \setminus \{P\}; \\ \dim \mathcal{I}_P/\mathfrak{m}_P\mathcal{I}_P\ (> 1) & \text{for } w = P. \end{cases}$$

Hence $[\mathcal{I}_P]_{\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)}$ (the completion of $\mathcal{I}_P$ in $\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$) is not equivalent to $[\mathcal{I}_{P'}]_{\mathcal{A}^2_{\alpha',\beta',\theta'}(\mathbb{B}^2)}$ (the completion of $\mathcal{I}'_P$ in $\mathcal{A}^2_{\alpha',\beta',\theta'}(\mathbb{B}^2)$) if $P \neq P'$. Now let us determine when two modules in the set

$$\{[\mathcal{I}_P]_{\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)} : -1 < \alpha, \beta, \theta < +\infty \text{ and } \alpha + \beta + \theta + 2 > 0\}.$$

are equivalent. In the following proposition, without loss of generality, we have assumed $P = 0$.

Proposition 4.2. *Suppose $\mathcal{I}$ is an ideal in $\mathbb{C}[z_1, z_2]$ with $V(\mathcal{I}) = \{0\}$. Then the Hilbert modules $[\mathcal{I}]_{\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)}$ and $[\mathcal{I}]_{\mathcal{A}^2_{\alpha',\beta',\theta'}(\mathbb{B}^2)}$ are unitarily equivalent if and only if $\alpha = \alpha', \beta = \beta'$ and $\theta = \theta'$.*

Proof. From the Hilbert Nullstellensatz, it follows that there exist an natural number N such that $\mathfrak{m}_0^N \subset \mathcal{I}$. Let $\mathcal{I}_{m,n}$ be the polynomial ideal generated by z_1^m and z_2^n. Combining (2.1) with Lemma 4.1 we see, in particular, that the submodules $[\mathcal{I}_{m,n}]_{\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)}$ and $[\mathcal{I}_{m,n}]_{\mathcal{A}^2_{\alpha',\beta',\theta'}(\mathbb{B}^2)}$ are unitarily equivalent for $m, n \geq N$. Let $K_{m,n}$ be the reproducing kernel for $[\mathcal{I}_{m,n}]_{\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)}$. We write $K_{\alpha,\beta,\theta}(z, w) = \sum_{i,j\geq 0} b_{ij} z_1^i z_2^j$ where

$$b_{ij} = \frac{\alpha + \beta + \theta + j + 2}{\alpha + \beta + \theta + 2} \cdot \frac{(\alpha + \theta + 2)_j}{(\theta + 1)_j} \cdot \frac{(\alpha + \beta + \theta + j + 3)_i}{i!}. \tag{4.1}$$

Let $I_{m,n} := \{(i,j) \in \mathbb{Z} \times \mathbb{Z} : i,j \geq 0, i \geq m \text{ or } j \geq n\}$. We note that

$$K_{m,n}(z,w) = \sum_{(i,j)\in I_{m,n}} b_{ij} z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j.$$

One easily see that the set $\{z_1^m, z_2^n\}$ forms a minimal set of generators for the sheaf corresponding to $[\mathcal{I}_{m,n}]_{\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)}$. The reproducing kernel then can be decomposed as

$$K_{m,n}(z,w) = \bar{w}_1^m K_1^{m,n}(z,w) + \bar{w}_2^n K_2^{m,n}(z,w) \text{ for some } r > 0 \text{ and } w \in \Delta(0;r).$$

Successive differentiation, using Leibnitz rule, gives

$$\begin{aligned} K_1^{m,n}(z,w)|_{w=0} &= \frac{1}{m!}\bar{\partial}_1^m K_{m,n}(\cdot,w)\}|_{w=(0,0)} = b_{m0} z_1^m \text{ and} \\ K_2^{m,n}(z,w)|_{w=0} &= \frac{1}{n!}\bar{\partial}_2^n K_{m,n}(\cdot,w)\}|_{w=(0,0)} = b_{0n} z_2^n. \end{aligned}$$

Therefore

$$s_1(\theta_1) \;=\; b_{m0} z_1^m + \theta_1 b_{0n} z_2^n,$$

where θ_1 denotes co-ordinate for the corresponding open chart in $\mathbb{P}^1$. Thus

$$\| s_1(\theta_1) \|^2 \;=\; b_{m0}^2 \| z_1^m \|^2 + b_{0n}^2 \| z_2^n \|^2 |\theta_1|^2 = \; b_{m0} + b_{0n}|\theta_1|^2.$$

Let $a_{m,n} = b_{0n}/b_{m0}$. Let $\mathcal{K}_{m,n}$ denote the curvature corresponding to the bundle $\mathcal{L}_{0,m,n}$ which is determined on the projective space $\mathbb{P}^1$ by the module $[\mathcal{I}_{m,n}]_{\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)}$. Thus we have

$$\begin{aligned} \mathcal{K}_{m,n}(\theta_1) &= \partial_{\theta_1}\partial_{\bar{\theta}_1} \ln \| s_1(\theta_1) \|^2 \;=\; \partial_{\theta_1}\partial_{\bar{\theta}_1} \ln(1 + a_{m,n}|\theta_1|^2) \\ &= \partial_{\theta_1} \frac{a_{m,n}\theta_1}{1 + a_{m,n}|\theta_1|^2} = \frac{a_{m,n}}{(1 + a_{m,n}|\theta_1|^2)^2}. \end{aligned}$$

Let $\mathcal{K}'_{m,n}$ denote the curvature corresponding to the bundle $\mathcal{L}'_{0,m,n}$ which is determined on the projective space $\mathbb{P}^1$ by the module $[\mathcal{I}_{m,n}]_{\mathcal{A}^2_{\alpha',\beta',\theta'}(\mathbb{B}^2)}$. As above we have

$$\mathcal{K}'_{m,n}(\theta_1) \;=\; \frac{a'_{m,n}}{(1 + a'_{m,n}|\theta_1|^2)^2}.$$

Since the submodules $[\mathcal{I}_{m,n}]_{\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)}$ and $[\mathcal{I}_{m,n}]_{\mathcal{A}^2_{\alpha',\beta',\theta'}(\mathbb{B}^2)}$ are unitarily equivalent, from Theorem 3.4, it follows that $\mathcal{K}_{m,n}(\theta_1) = \mathcal{K}'_{m,n}(\theta_1)$ for θ_1 in an open chart $\mathbb{P}^1$ and $m,n \geq N$. Thus

$$\frac{a_{m,n}}{(1 + a_{m,n}|\theta_1|^2)^2} \;=\; \frac{a'_{m,n}}{(1 + a'_{m,n}|\theta_1|^2)^2}.$$

This shows that $(a_{m,n} - a'_{m,n})(1 + a_{m,n}a'_{m,n}|\theta_1|^2) = 0$. So $a_{m,n} = a'_{m,n}$ and hence

$$\frac{b_{0n}}{b_{m0}} = \frac{b'_{0n}}{b'_{m0}} \tag{4.2}$$

for all $m,n \geq N$. This also follows from the equation (3.2). It is enough to consider the cases $(m,n) = (N,N), (N,N+1), (N,N+2)$ and $(N+1,N)$ to prove the Proposition. From equation (4.2), we have

$$\frac{b_{(N+1)0}}{b_{N0}} = \frac{b'_{(N+1)0}}{b'_{N0}},\; \frac{b_{0(N+1)}}{b_{0N}} = \frac{b'_{0(N+1)}}{b'_{0N}} \text{ and } \frac{b_{0(N+2)}}{b_{0(N+1)}} = \frac{b'_{0(N+2)}}{b'_{0(N+1)}}. \tag{4.3}$$

Let $A = \alpha + \beta + \theta, B = \alpha + \theta$ and $C = \theta$. From equation (4.1), we have

$$\frac{b_{(N+1)0}}{b_{N0}} = \frac{A + N + 3}{N + 1},\; \frac{b_{0(N+1)}}{b_{0N}} \;=\; \frac{A + N + 3}{A + N + 2} \cdot \frac{B + N + 2}{C + N + 1}$$

and

$$\frac{b_{0(N+2)}}{b_{0(N+1)}} = \frac{A+N+4}{A+N+3} \cdot \frac{B+N+3}{C+N+2}.$$

From (4.3), it follows that $A = A'$ and

$$BC' + B(N+1) + C'(N+2) = B'C + B'(N+1) + C(N+2), \tag{4.4}$$

$$BC' + B(N+2) + C'(N+3) = B'C + B'(N+2) + C(N+3). \tag{4.5}$$

Subtracting (4.5) from (4.4), we get $B - C = B' - C'$ and thus $\theta = \theta'$. Therefore $\frac{b_{0(N+1)}}{b_{0N}} = \frac{b'_{0(N+1)}}{b'_{0N}}$ implying $B = B'$ and hence $\alpha = \alpha'$. Lastly $A = A'$ and in consequence $\beta = \beta'$. □

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